ALGEBRAIC DERIVATIONS IN THE FIELD OF MIKUSIŃSKI'S OPERATORS

MORISUKE HASUMI

(Received August 5, 1961)

Let $\mathfrak{C}$ be the set of complex-valued functions in R^1, continuous in $0 \leqq t < +\infty$ and vanishing identically in $t < 0$. Under pointwise addition and convolution as multiplication, $\mathfrak{C}$ forms a commutative algebra over the complex numbers which contains no zero divisor. Elements of the quotient field $\mathfrak{Q}$ of the ring $\mathfrak{C}$ are the operators of Mikusiński. Every function $f(t)$ in $\mathfrak{C}$ is regarded as an element of $\mathfrak{Q}$ which is denoted by $\{f\}$. Recently, Mikusiński [3] [4] has discussed some properties of an algebraic derivation D in $\mathfrak{Q}$ which is defined by $Du = \{-tu(t)\}$ for any $u \in \mathfrak{C}$. In any algebra A, an algebraic derivation F is defined as a linear mapping of A into some algebra B, which contains A as a subalgebra, such that $F(ab) = F(a)b + aF(b)$ for any $a,b \in A$. Here we shall study algebraic derivations in the field $\mathfrak{Q}$ which are continuous in some sense specified below. Since any derivation in $\mathfrak{Q}$ is completely determined by its effects on the ring $\mathfrak{C}$, we have only to consider algebraic derivations from $\mathfrak{C}$ into $\mathfrak{Q}$. Notations are the same as in Mikusiński [3], unless otherwise stated.

Clearly, $\mathfrak{C}$ is a locally convex space with respect to the topology of uniform convergence on compact sets in R^1. Moreover, it is a locally multiplicatively-convex F-algebra in the sense of Michael [2]. A sequence $\{a_n : n = 1, 2, \dots\}$ in $\mathfrak{Q}$ is said to be convergent to an element $a \in \mathfrak{Q}$ if there exists an element $b \in \mathfrak{Q}$ such that ba_n and ba are contained in $\mathfrak{C}$ and $ba_n \to ba$ with respect to the topology of $\mathfrak{C}$. Then any sequence has at most one limit. The pseudo-topology of $\mathfrak{Q}$ thus defined is used in what follows. It is known that there is no locally convex Hausdorff topology in $\mathfrak{Q}$ which induces this notion of convergence for sequences (cf. [1], [3]).

THEOREM 1. *A linear mapping F of $\mathfrak{C}$ into $\mathfrak{Q}$ is a continuous derivation if and only if there exists an element $a \in \mathfrak{Q}$ such that $F = aD$, i.e. $F(u) = a \cdot D(u)$ for any $u \in \mathfrak{C}$, where $D(u) = \{-tu(t)\}$.*

PROOF. Let F be a continuous derivation of $\mathfrak{C}$ into $\mathfrak{Q}$. Since $\{t^k\} = k!\, l^{k+1} (k = 0, 1, 2, \dots)$, any polynomial $\{f(t)\} = \left\{\sum_{k=0}^{n} \alpha_k t^k\right\}$ is expressed as