ON THE SATURATION AND BEST APPROXIMATION

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(Received April 16, 1962)

Let $f(x)$ be an integrable function with period 2π and let its Fourier series be

$$a_0/2+\sum_{k=1}^{\infty}(a_k\cos kx+b_k\sin kx)=\sum_{k=0}^{\infty}A_k(x).$$

Denote the method of typical means of this series by

$$R_n^{\lambda}(f)=\sum_{k=0}^{n-1}\left(1-\frac{k^{\lambda}}{n^{\lambda}}\right)A_k(x).$$

Then this method saturates with the order $n^{-\lambda}$, that is, we have

THEOREM A. *For the typical means,*

(1°) $f-R_n^{\lambda}(f)=o(n^{-\lambda})\Longleftrightarrow f=a$ constant,

(2°) $f-R_n^{\lambda}(f)=O(n^{-\lambda})\Longleftrightarrow f\in W^{\lambda}$,

where W^{λ} means the class of functions for which

$$\sum_{k=1}^{\infty}k^{\lambda}A_k(x)\sim f^{\lambda}\in L^{\infty}(0,2\pi).$$

See Aljančić [1], Sunouchi [3] Sunouchi-Watari [4]. Recently Aljančić [2] proved the following theorm.

THEOREM B. *Let $k=0,1,\ldots\ldots$ and $0<\alpha\leqq 1$. Then*

$$f^{(k)}(x)\in {}^2\Lambda_{\alpha}(k+\alpha<\lambda)\Longleftrightarrow f-R_n^{\lambda}(f)=O(n^{-k-\alpha}),$$

where $f^{(k)}(x)\in {}^2\Lambda_{\alpha}$ means

$$f^{(k)}(x+h)+f^{(k)}(x-h)-2f^{(k)}(x)=O(|h|^{\alpha}).$$

However this fact is not confined to only the typical means, but also is valid for more general approximation processes. Indeed we can deduce Theorem B from Theorem A by method of the moving average.

THEOREM. *Let $k=0,1,2,\ldots\ldots$, and $0<\alpha\leqq 1$. Suppose that for linear approximation processes $T_n(f)$*

(1°) $|f(x)|\leqq M_1$ *implies* $|T_n(f)(x)|\leqq k_1M_1$,

and