Tôhoku Math. Journ.
20(1968), 443-497.

RATIONALITY PROPERTIES OF LINEAR ALGEBRAIC GROUPS II

A. BOREL AND T. A. SPRINGER

(Received March 5, 1968)

This paper is a development of [4], and gives a more detailed treatment of the topic named in the title. It includes in particular the birational equivalence with affine space, over the groundfield, of the variety of Cartan subgroups of a k-group G, the splitting of G over a separable extension of k if G is reductive, some results on unipotent groups operated upon by tori, and on the existence of subgroups of G whose Lie algebra contains a given nilpotent element of the Lie algebra $\mathfrak{g}$ of G.

Discussing as it does a number of known results (due mostly to Rosenlicht and Grothendieck), this paper is to be viewed as partly expository. In fact, besides proving some new results, our main goal is to provide a rather comprehensive, albeit not exhaustive, account of our topic, from the point of view sketched in [4].

Our basic tools are some rationality properties of transversal intersections and of separable mappings, the Jordan decomposition in $\mathfrak{g}$, and purely inseparable isogenies of height one. They are reviewed or discussed in section 1.13, §3 and §5 respectively. Thus Lie algebras of algebraic groups play an important role in this paper and, for the sake of completeness, we have collected in §1 a number of definitions and facts pertaining to them.

§2 reproves a result of Grothendieck ([12], Exp. XIV) stating that $\mathfrak{g}$ is the union of the subalgebras of its Borel subgroups. Its main use for us is to reduce to Lie algebras of solvable groups the existence proof of the Jordan decomposition.

§4 discusses subalgebras $\mathfrak{s}$ of $\mathfrak{g}$ consisting of semi-simple elements, to be called "toral subalgebras" of $\mathfrak{g}$. They are tangent to maximal tori, and have several properties similar to that of tori in G, in particular: the centralizer $Z(\mathfrak{s})=\{g \in G, \mathrm{Ad}g(X)=X(X \in \mathfrak{s})\}$ of $\mathfrak{s}$ in G is defined over k if $\mathfrak{s}$ is, (see 4.3 for $Z(\mathfrak{s})^0$, 6.14 for $Z(\mathfrak{s})$), its Lie algebra is $\mathfrak{z}(\mathfrak{s}) = \{X \in \mathfrak{g}, [\mathfrak{s}, X] = 0\}$. If $\mathfrak{s}$ is spanned by one element X, the conjugacy class of X is isomorphic to $G/Z(\mathfrak{s})$. This paragraph also gives some conditions under which a subalgebra of $\mathfrak{g}$ is algebraic, and reproves some results of Chevalley [8] in characteristic zero.

§6 introduces regular elements, Cartan subalgebras in $\mathfrak{g}$, and the subgroups of type (C) of ([12], Exp. XIII) in G. By definition here, a Cartan subalgebra