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POSITIVE SELFADJOINT EXTENSIONS OF POSITIVE SYMMETRIC OPERATORS

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1. Introduction. Positive selfadjoint extensions of a symmetric operator have been investigated by many mathematicians: J. von Neumann, K. Friedrichs, M. Krein, M. Birman and others. Especially M. Krein [3] observed the class of all positive selfadjoint extensions of a given positive symmetric operator, and proved among others that, in case of a densely defined operator, the greatest and the smallest positive selfadjoint extension exist. The greatest extension is shown to coincide with the extension, established by Friedrichs, while the smallest one coincides with the extension, considered by von Neumann in case of a strongly positive operator.

In this paper, starting with the well known theorem of Friedrichs, we shall investigate the structure of the greatest extension (Friedrichs extension) T_μ and the smallest one (von Neumann extension) T_M of a given positive symmetric operator T from various points of view. Theorem 1 gives a necessary and sufficient condition for T with non-dense domain to admit positive selfadjoint extensions. If any one of such extensions exists, the von Neumann extension is shown to exist, and its domain is explicitly determined. Among many consequences of this theorem is a simple description of the von Neumann extension, when it is bounded (Theorem 2). In contrast with the identity: $(T+a)_\mu - a = T_\mu$ for all positive number a, $(T+a)_M - a$ varies largely according to a. Theorem 3 shows that $(T+a)_M - a$ converges, in a natural sense, to the von Neumann extension T_M or to the Friedrichs extension T_μ according as $a \to 0$ or $\to \infty$. This theorem permits us to determine the spectrum of T_M or T_μ, when T_M is compact or T_μ has compact resolvent.

2. Preliminaries. A linear operator T on a Hilbert space is, by definition, *symmetric* if

$$(Tf, g) = (f, Tg) \qquad (f, g \in \boldsymbol{D}(T));$$

here the domain $\boldsymbol{D}(T)$ is not assumed to be dense. T is called *positive* (resp. *strongly positive*), if $(Tf,f) \geqq 0$ (resp. $\geqq \varepsilon(f,f)$ for a constant $\varepsilon > 0$).

A positive selfadjoint operator S_1 is called *greater* than another S_2, or the