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REMARKS ON THE RIESZ DECOMPOSITION FOR SUPERMARTINGALES

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In this paper we shall give an another proof of the Riesz decomposition theorem for supermartingales and we shall consider on the Riesz-type decomposition for local supermartingales.

1. Let $(\Omega, \mathfrak{F}, P)$ be the basic P-complete probability space and let $\mathfrak{F}_n$ be a sub σ-field of $\mathfrak{F}$ such that $\mathfrak{F}_m \subset \mathfrak{F}_n$ whenever $m < n$. It is clear that $E[x_n]$ decreases if $(x_n, \mathfrak{F}_n)$ is a supermartingale. We assume here the integrability of x_n for each n.

THEOREM 1. *Let $(x_n, \mathfrak{F}_n)$ be a supermartingale. Then x_n can be written as*

$$x_n = x_n^* + y_n$$

where $(x_n^*, \mathfrak{F}_n)$ is a martingale and $(y_n, \mathfrak{F}_n)$ is a positive supermartingale if and only if

$$\text{(A)} \qquad \inf_n E[x_n] > -\infty$$

(*there is no uniqueness*)

PROOF. The condition is obviously necessary. Let us prove the sufficiency. Since $(x_n, \mathfrak{F}_n)$ is a supermartingale, we have

$$E[x_{n+k+1} | \mathfrak{F}_n] \leqq E[x_{n+k} | \mathfrak{F}_n] \leqq x_n \, .$$

Put for each n

$$x_n^* = \lim_{k\to\infty} E[x_{n+k} | \mathfrak{F}_n] \, .$$

Clearly $x_n - x_n^* \geqq 0$ and x_n^* is $\mathfrak{F}_n$-measurable. If the condition (A) is fulfilled, then from the monotone convergence theorem we have