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ON THE LAW OF THE ITERATED LOGARITHM FOR LACUNARY TRIGONOMETRIC SERIES

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Introduction. Throughout this note we set

$$S_N(x) = \sum_{k=1}^{N} a_k \cos 2\pi(n_k x + \alpha_k) \quad \text{and} \quad A_N = \left(2^{-1} \sum_{k=1}^{N} a_k^2\right)^{1/2},$$

where $\{n_k\}$ is a sequence of positive integers and we assume that

$$A_N \to +\infty\ ; \qquad \text{as } N \to +\infty\ . \tag{1.1}$$

In [2] M. Weiss has proved the following

THEOREM. *If $\{n_k\}$ and $\{a_k\}$ satisfy the conditions*

$$n_{k+1}/n_k > 1 + c\ , \qquad \textit{for some } c > 0\ , \tag{1.2}$$

and

$$a_N = o(\sqrt{A_N^2/\log\log A_N})\ , \qquad \textit{as } N \to +\infty\ , \tag{1.3}$$

then we have, for any sequence of real numbers $\{\alpha_k\}$,

$$\overline{\lim_{N\to\infty}}\, (2A_N^2 \log\log A_N)^{-1/2} S_N(x) = 1\ , \qquad \text{a.e.}\ .$$

That is, the same law of the iterated logarithm holds for

$$\{\cos 2\pi(n_k x + \alpha_k)\}$$

as for the sequence of normalized, uniformly bounded independent random variables with vanishing mean values.

The purpose of the present note is to weaken the *lacunarity* condition (1.2). But we could show only the inequality "$\overline{\lim} \leqq 1$". In fact we prove the following

THEOREM. *Let $\{n_k\}$ and $\{a_k\}$ satisfy the conditions*

$$n_{k+1}/n_k > 1 + c\,k^{-\alpha}\ , \qquad \textit{for some } c > 0 \textit{ and } 0 < \alpha \leqq 1/2\ , \tag{1.4}$$

and

$$a_N = O(\sqrt{A_N^2/N^{2\alpha}(\log A_N)^{1+\varepsilon}})\ , \qquad \textit{for some } \varepsilon > 0,\ \textit{as } N \to +\infty\ . \tag{1.5}$$