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ERGODIC THEOREMS FOR SEMI-GROUPS IN L_p, $1 < p < \infty$

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1. **Introduction.** In what follows we shall assume p fixed, $1 < p < \infty$. Let $(X, \mathcal{M}, m)$ be a σ-finite measure space and let $\{T_t; t \geqq 0\}$ be a semi-group of positive linear operators in $L_p(X) = L_p(X, \mathcal{M}, m)$ which is strongly integrable over every finite interval. It is then known (cf. [2], p. 686) that for each $f \in L_p(X)$ there exists a scalar function $T_t f(X)$, measurable with respect to the product of Lebesgue measure and m, such that for almost all t, $T_t f(x)$ belongs to the equivalence class of $T_t f$. Moreover there exists a set $E(f)$ with $m(E(f)) = 0$, dependent on f but independent of t, such that if $x \notin E(f)$ then $T_t f(x)$ is integrable on every finite interval $[a, b]$ and the integral $\int_a^b T_t f(x) dt$, as a function of x, belongs to the equivalence class of $\int_a^b T_t f dt$. We write $S_a^b f(x)$ for $\int_a^b T_t f(x) dt$. The purpose of this note is to investigate the almost everywhere convergence of $S_0^b f(x)/S_0^b g(x)$ and $S_0^b f(x)/b$ as $b \uparrow \infty$.

2. **Preliminaries.** If $A \in \mathcal{M}$ then $L_p(A)$ denotes the Banach space of all $L_p(X)$-functions that vanish a.e. on $X - A$. A set $A \in \mathcal{M}$ is called *closed* under a positive linear operator T on $L_p(X)$ if $f \in L_p(A)$ implies $Tf \in L_p(A)$. The adjoint operator of T is denoted by T^*.

PROPOSITION. *If T is a positive linear operator on $L_p(X)$ such that $\sup_n \| (1/n) \sum_{k=0}^{n-1} T^k \|_p < \infty$ and $\lim_n \| (1/n) T^n f \|_p = 0$ for every $f \in L_p(X)$, then the space X uniquely decomposes into two measurable sets Y and Z such that*

(a) *Z is closed under T,*

(b) *if $f \in L_p(Z)$ then $\lim_n \| (1/n) \sum_{k=0}^{n-1} T^k f \|_p = 0$,*

(c) *there exists a nonnegative function u in $L_q(Y)$ such that $u > 0$ a.e. on Y and $T^* u = u$, where $q = p/(p-1)$.*

PROOF. We may choose a nonnegative function u in $L_q(X)$ such that $T^* u = u$ and if $0 \leqq v \in L_q(X)$ is invariant under T^* then $\operatorname{supp} v \subset \operatorname{supp} u$. Let $Y = \operatorname{supp} u$ and $Z = X - Y$. Since $T^* u = u$, (a) is obvious. To see (b), let $0 \leqq g \in L_p(Z)$. Then the mean ergodic theorem ([2], p. 661) implies that strong-$\lim_n (1/n) \sum_{k=0}^{n-1} T^k g = g_0$ for some $0 \leqq g_0 \in L_p(Z)$ with $Tg_0 = g_0$. Here