Tôhoku Math. Journ.
26 (1974), 285-303.

APPROXIMATION OPERATORS ON BANACH SPACES OF DISTRIBUTIONS

R. GOPALAN†

(Received April 9, 1973)

Abstract. An approximation process $\{\Gamma_n\}_{n\in P}$ on a Banach subspace X of $\mathscr{X}'$ [Zemanian A. H. [36]], satisfying either a Jackson type inequality or a Bernstein type inequality of order $\rho(n)$ on X with respect to Y of X, is being related to a class of Banach subspaces $\{X_\lambda\}_{\lambda\in J}$ of $\mathscr{X}'$, on each of which, $\{\Gamma_n\}_{n\in P}$ defines a sequence of multiplier type operators, satisfying the same inequality with same order. Sufficient conditions for $X_\lambda\subset\mathscr{X}'$, $\lambda\in J$ are given. Results are illustrated by examples.

1. Introduction. For a Banach space X, a sequence $\{\Gamma_n\}_{n\in P}$ of bounded linear operators $\Gamma_n: X\to X$, with $P=\{1,2,3,\cdots\}$ is called an approximation process on X, if $\Gamma_n f\to f$ in X $\forall f\in X$. For suitable subspaces Y, Λ of X (Λ being fixed, $\dim(\Lambda)<\infty$) and function $\rho(n)\geqq 0$, $\rho(n)\searrow 0$ on P, an approximation process $\{\Gamma_n\}$ on X is said to,

(I) satisfy a Jackson-type inequality of order $\rho(n)$ on X with respect to Y, if $\forall f\in Y$, $\|\Gamma_n f-f\|_X\leqq C\rho(n)\|f\|_Y$;

(II) satisfy a Bernstein type inequality of order $\rho(n)$ on X with respect to Y, if $\bigcup_{n\in P}\Gamma_n(X)\subset Y$ and $\forall f\in X$, $\|\Gamma_n f\|_Y\leqq C_1(\rho(n))^{-1}\|f\|_X$. ($C$, C_1 constants independent of n);

(III) be saturated with order $\rho(n)$ on X with saturation class Y, if for $f\in X$, $\|\Gamma_n f-f\|_X=\begin{cases}o(\rho(n))\Leftrightarrow f\in\Lambda\\ O(\rho(n))\Leftrightarrow f\in Y, Y-\Lambda\neq\varnothing.\end{cases}$

For such $\{\Gamma_n\}$ as in (III) above, the inverse problem is the characterization of elements of the sets

$\{f\in X\,|\,\|\Gamma_n f-f\|_X=O(\eta(n))\}$ with some $\eta(n)\geqq 0$, $\eta(n)\searrow 0$, $\dfrac{\rho(n)}{\eta(n)}\to 0$ as $n\to\infty$.

Given a Banach subspace X of a certain space $\mathscr{X}'$ of generalized functions, each $f\in\mathscr{X}'$ having Fourier expansion with respect to an orthonormal system $\{\psi_n\}_{n\in N}$ ($N=0,1,2,3,\cdots$) and given an approximation process $\{\Gamma_n\}_{n\in P}$ related to $\{\psi_n\}_{n\in N}$ on X, satisfying (J) Jackson-type inequality or (B) Bernstein-type inequality or for X, having (S) saturation and inverse theorems, the aim of this paper is to determine a family of related

† Isaak Killam Doctoral Fellow 1971-73.
Results of this paper are part of the author's doctoral dissertation written under the direction of Professor Z. Ditzian at University of Alberta.