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ON ARTIN L-FUNCTIONS

KÔJI UCHIDA

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Let k be an algebraic number field of finite degree. Let K be a Galois extension of k of finite degree. Let G be the Galois group of this extension. Let χ be a character of G. Then Artin L-function $L(s, \chi)$ is defined. For some groups G, $L(s, \chi)$ is known to be an entire function for every non-trivial irreducible character χ [2, p. 225]. These cases can be proved through Blichfeldt's theorem [3, p. 348] reducing to abelian cases, i.e., Hecke L-functions. This theorem can be applied for other groups, i.e., for supersolvable groups. A group G is called supersolvable if G has normal subgroups $H_0, H_1, \cdots, H_r$ such that $G = H_0 \supset H_1 \supset \cdots \supset H_r = \{e\}$ and every H_{i-1}/H_i is cyclic [4].

THEOREM 1. *If the Galois group G is supersolvable, $L(s, \chi)$ is entire for every non-trivial irreducible character χ.*

PROOF.[1] If G is abelian, $L(s, \chi)$ is a Hecke L-function which is entire. So we assume that G is not abelian and we will prove by induction on the order of G. Let χ be the character of a representation module (G, V). If there exists a non-trivial normal subgroup N which operates trivially on V, χ is a character of G/N. As G/N is also supersolvable, $L(s, \chi)$ is entire by induction. Now we assume that there exists no such normal subgroup. Then G is a subgroup of $GL(V)$. Let C be the center of G. As G/C is also supersolvable, there exists a normal subgroup A of G such that A/C is cyclic and $A \neq C$. Then A is abelian because C is in the center of A. Now Blichfeldt's theorem shows that there exists a proper subgroup H of G such that $\chi = \varphi^G$ for some character φ of H, where φ^G means the induced character of G. It is easy to see that φ is non-trivial and irreducible. As $L(s, \chi) = L(s, \varphi)$ and as H is also supersolvable, our assertion is proved by induction.

REMARK. Professor M. Ishida kindly suggested this proof when G is nilpotent. We note that every finite nilpotent group is supersolvable.

[1] This proof shows that $L(s, \chi)$ is entire for every χ if the Galois group is an M-group. Hence Theorem 1 is a special case of Huppert's Theorem [5, p. 580]. We also note that every M-group is solvable [5, p. 581].