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APPROXIMATION BY OSCILLATING GENERALIZED POLYNOMIALS

R. A. BELL AND S. M. SHAH

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1. Introduction. Oscillating generalized polynomials extend to generalized polynomials the concept of oscillating polynomials (defined below) which were studied first by Bernstein ([3]; [4]). The subjects of oscillating generalized polynomials (abbreviated hereafter as *OGP*'s), and uniform approximations, by polynomials with real coefficients, to real powers of x are closely related. Indeed if r_i is a positive real number for $i = 1, 2, \cdots, k$ such that for some integer n_0, $n_0 < r_1 < \cdots < r_k < n_0 + 1$, then $q(x)$ is the best approximation on $[0, 1]$ to $\sum_{i=1}^{k} x^{r_i}$ by a polynomial of degree n if and only if $\sum_{i=1}^{k} x^{r_i} - q(x)$ is an *OGP*.

In Section 2 we develop the theory of *OGP*'s. We prove an existence and uniqueness theorem. Further we derive properties of *OGP*'s useful in approximations to real powers of x. In particular we show that if $p(x) = \sum_{k=0}^{n} A_k g_{\alpha_k}(x)$ and $q(x) = \sum_{k=0}^{n} B_k g_{\alpha_k}(x)$ are distinct generalized polynomials (abbreviated hereafter as *GP*'s) where $A_k = B_k$ for at least one k with g_{α_k} not a constant function and p is an *OGP*, then $\max_{0 \leq x \leq 1} |q(x)| \equiv \| q \| > \max_{0 \leq x \leq 1} |p(x)| \equiv \| p \|$.

In Section 3 we study, by use of the theory of *OGP*'s, the uniform approximation in $[0, 1]$ of real powers of x by polynomials with real coefficients. Here we derive lower bounds for the best approximation error in $[0, 1]$ to x^{α}, where α is a real number lying in $(0, 1/3)$, by polynomials of a given degree. Further, we give in Examples 4 and 5 the polynomials which provide the best uniform approximation to $x^{1/\pi}$ and $1/2(x^{1/3} + x^{1/2})$, respectively, by polynomials of degree not exceeding n.

2. Oscillating generalized polynomials. Throughout this paper n, $\alpha_0, \cdots, \alpha_n$ will denote integers such that $n \geqq 1$ and $0 \leqq \alpha_0 < \alpha_1 < \cdots < \alpha_n$. We now define *OGP*'s.

DEFINITION 2.1. Let $\{g_\alpha\}_{\alpha=0}^{\infty}$ be a sequence of functions, real valued, non-negative and continuous on $[0, 1]$ and analytic on $(0, 1]$. Further suppose that g_α is not a constant function if $\alpha \geqq 1$, g_0 is not identically zero and $g_\alpha(0) = 0$ unless g_α is a constant. Then $\{g_\alpha\}_{\alpha=0}^{\infty}$ is said to have property $\mathscr{D}$ if and only if the following hold: