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STABILITY OF BIFURCATING SOLUTIONS OF THE GIERER-MEINHARDT SYSTEM

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Introduction. In this paper we are concerned with stationary solutions of a system of semilinear parabolic partial differential equations arising in the biological pattern formation theory. A most fundamental problem in morphogenesis is to explain how the initially almost homogeneous state of cells (or tissues) gains spatial inhomogeneity or spatial patterns. In his pioneering paper [10], Turing considered this problem by introducing the model governed by a system of (linear) ordinary differential equations. Developing Turing's idea, Gierer and Meinhardt [4] proposed a reaction-diffusion system of the following type:

$$\text{(G-M)}\qquad \begin{cases} \dfrac{\partial a}{\partial s} = D_a \dfrac{\partial^2 a}{\partial y^2} - \tilde{\mu} a + c\tilde{\rho}\dfrac{a^2}{h} + \rho_0 \tilde{\rho}\,, \\ \dfrac{\partial h}{\partial s} = D_h \dfrac{\partial^2 h}{\partial y^2} - \nu h + c'\rho' a^2\,, \end{cases} \qquad \text{for} \quad s > 0\,, \quad 0 < y < \tilde{L}\,.$$

Here, D_a, D_h, $\tilde{\mu}$, ν, c, c', $\tilde{\rho}$ and ρ' are all positive constants and ρ_0 is a non-negative constant. The positive functions $a(s, y)$ and $h(s, y)$ represent the concentrations of an activator and an inhibitor, respectively.

From a biological point of view, a natural boundary condition is the zero flux condition, i.e.,

$$\partial a/\partial y = \partial h/\partial y = 0 \quad \text{at both end points of the interval}\,.$$

Note that the system (G-M) has a unique constant stationary solution subject to this boundary condition.

Under suitably chosen values of the constants D_a, D_h, $\cdots$, ρ_0, numerical analyses show that the solutions of (G-M) corresponding to almost constant initial values tend to the stationary solutions exhibiting spatial wavy patterns. (This suggests that the constant solution is unstable.) The place where the activator highly concentrates is regarded as the position at which cell differentiation or division begins.

It is also predicted numerically that the wave length of the station-