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A FIXED POINT THEOREM AND ITS APPLICATION IN ERGODIC THEORY

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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The purpose of this paper is to prove a simple fixed point theorem in Banach spaces, and to show its application in ergodic theory. The theorem asserts the existence of a unique fixed point for affine transformations and the convergence of successive approximations to the fixed point. In the special case of linear operators in L^1 generated by point-to-point nonsingular transformations, this fixed point theorem demonstrates the existence and uniqueness of invariant measures and the exactness of corresponding measurable dynamical systems. The theorem thus gives a new tool for proving the exactness of some measurable endomorphisms.

The paper is divided into four parts. In Section 1 an abstract version of the fixed point theorem is proved. From the formal point of view it remembles some known results of Edelstein [1]. The proof, however, is based on ideas due to Pianigiani and Yorke [7]. Section 2 contains the specialization of the fixed point theorem to the space L^1. In Section 3 the general theory is examined in the case of expanding mappings of differentiable manifolds and a new simpler proof of the well known Krzyżewski-Szlenk theorem [5] is presented. In the proof once again the ideas of Pianigiani and Yorke are used. Finally, Section 4 is devoted to the study of a class of dynamical systems generated by piecewise convex transformations.

1. Fixed point theorem. Let E, $\| \ \|$ be a Banach space. A closed convex set $C \subset E$ is said to be imbedded in V ($V \subset E$) if for each two different points $x_1, x_2 \in C$ the closed interval $[0, 1]$ is contained in the interior of the set $\{\lambda \in R: \lambda x_1 + (1-\lambda)x_2 \in V\}$. The distance between a nonempty set $C \subset E$ and a point $x \in E$ is defined, as usual by

$$\rho(x, C) = \inf \{\|x - y\|: y \in C\}\ .$$

A sequence $\{x_n\} \subset E$ converges to C $(x_n \to C)$ if $\lim_n \rho(x_n, C) = 0$. In particular $x_n \to x_0$ always stands for $\|x_n - x_0\| \to 0$.