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DECOMPOSITION THEOREM FOR PROPER KÄHLER MORPHISMS

MORIHIKO SAITO

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Introduction. In [S1], [S2] we introduced the notion of *polarizable Hodge Modules* on complex analytic spaces, which corresponds philosophically to that of pure perverse sheaves in characteristic p [BBD]. If X is smooth, $\mathrm{MH}(X, n)^p$ the category of polarizable Hodge Modules of weight n (and with $\boldsymbol{k}$-structure) is a full subcategory of the category of filtered holonomic $\mathscr{D}_X$-Modules (M, F) with $\boldsymbol{k}$-structure by a given isomorphism $\alpha: \mathrm{DR}(M) \simeq \boldsymbol{C} \otimes_{\boldsymbol{k}} K$ for a perverse sheaf K (defined over $\boldsymbol{k}$). Here $\boldsymbol{k}$ is a subfield of $\boldsymbol{R}$, and we assume for simplicity $\boldsymbol{k} = \boldsymbol{R}$ in this note. In general $\mathrm{MH}(X, n)^p$ is defined using local embeddings into smooth varieties, and the underlying perverse sheaves K are globally well-defined. We can show that the category $\mathrm{MH}(X, n)^p$ is a *semi-simple abelian* category, and admits the *strict support decomposition*:

$$\mathrm{MH}(X, n)^p = \bigoplus_Z \mathrm{MH}_Z(X, n)^p \qquad \text{locally finite on} \quad X, \tag{0.1}$$

where Z are closed irreducible subspaces of X, and $\mathrm{MH}_Z(X, n)^p$ is the full subcategory of $\mathrm{MH}(X, n)^p$ with *strict support* Z, i.e. the underlying perverse sheaves of its objects are intersection complexes with local system coefficients, and supported on Z (or $\varnothing$). This decomposition is *unique*, because there is no nontrivial morphism between the Hodge Modules with different strict supports. The category $\mathrm{MH}_Z(X, n)^p$ depends only on Z and n (independent of X), and we have the equivalence of categories [S5]:

$$\mathrm{MH}_Z(X, n)^p \simeq \mathrm{VSH}(Z, n - \dim Z)^p_{\mathrm{gen}} \tag{0.2}$$

where the right hand side is the category of polarizable variations of $\boldsymbol{R}$-Hodge structures of weight $n - \dim Z$ defined on Zariski-open dense smooth subsets of Z, and the polarizations on Hodge Modules correspond bijectively to those of variations of Hodge structures. The main result of [S1], [S2] was the relative version of the Kähler package:

(0.3) THEOREM. *Let $f: X \to Y$ be a cohomologically projective morphism of complex analytic spaces, i.e. there is $l \in H^2(X, \boldsymbol{R}(1))$ which is locally on Y the pull-back of a multiple of the hyperplane section class by $X \subsetneq Y \times \boldsymbol{P}^m$. Then we have the natural functors*:

$$\mathscr{H}^j f_*: \mathrm{MH}(X, n)^p \to \mathrm{MH}(Y, n+j)^p \tag{0.3.1}$$

compatible with the corresponding functors ${}^p\mathscr{H}^j f_*$ on the underlying perverse sheaves [BBD], *and the relative hard Lefschetz*: