Tôhoku Math. J.
46 (1994), 557–572

AN ALGORITHMIC DESINGULARIZATION OF 3-DIMENSIONAL TORIC VARIETIES

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(Received August 27, 1993, revised May 26, 1994)

Abstract. We present an algorithmic procedure to desingularize every 3-dimensional toric variety, while keeping under control the Euler characteristic of the varieties computed during the process. We prove that our upper bounds for the Euler characteristic of the desingularized toric varieties are the best possible.

Introduction. We refer to [4] and [5] for resolutions of singularities of toric varieties in general. In [3, p. 48] and in [2, 8.2] it is explained how to get a simplicial subdivision of a general nonsimplicial fan.

We present a method to desingularize every 3-dimensional toric variety $X=X_{\Delta}$ (with Δ a simplicial fan): a subdivision $\nabla(\Delta)$ of Δ is constructed by successively starring each 3-dimensional cone $\sigma\in\Delta$ of multiplicity $\mathrm{mult}(\sigma)>1$ at a nonzero primitive vector $p_{\sigma}\in\sigma\cap\boldsymbol{Z}^3$ such that the sum of the multiplicities of the cones obtained by the starring is the smallest possible. Here, as usual, $\mathrm{mult}(\sigma)$ is the index of the subgroup of $\boldsymbol{Z}^3$ generated by the primitive (integral) generators of σ.

When applied to the 2-dimensional case, our method yields the familiar construction of the coarsest nonsingular subdivision of Δ: this is uniquely determined by the set S of integral points of the compact faces of the boundary of the convex hull of the set of integral nonzero points in each cone of Δ. As is well known, for every dimension >2, S generally contains too few points, and these are not very useful to construct desingularizations.

In the 3-dimensional case we rather focus attention on the set S' of $\leq_{\sigma}$-minimal integral vectors of σ with respect to the order induced by each cone $\sigma\in\Delta$. As shown in our paper, S' is the set of primitive generators of those rays which belong to *every* nonsingular subdivision of Δ. While in the 2-dimensional case $S=S'$, in the 3-dimensional case S' is strictly larger than S, and is a key tool in our desingularization algorithm.

Our method also keeps under control the number of cones obtained during the desingularization process and yields, for every 3-dimensional compact toric variety $X=X_{\Delta}$ (where each 3-dimensional cone $\sigma\in\Delta$ is simplicial) a desingularization X' of X such that $E(X')\leq -E(X)+2\sum_{\sigma\in\Delta^{(3)}}\mathrm{mult}(\sigma)$. Here, $E(X)$ is the Euler characteristics, which coincides with the number $\sharp\Delta^{(3)}$ of 3-dimensional cones in Δ. We show that our upper bound is the best possible.

1991 *Mathematics Subject Classification*. Primary 14M25; Secondary 11H06, 52C07.