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ON THE SOLUTIONS OF THUE EQUATIONS

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Abstract. Silverman's estimate for the number of integral points of the so-called Thue equation is improved in a certain special case. A sufficient condition for the non-existence of rational solutions is also given.

Introduction. Let $k/\boldsymbol{Q}$ be a finite extension, $p(X, Y) \in k[X, Y]$ a homogeneous polynomial of degree $n \geq 3$ with non-zero discriminant, and $a \in k^{\times} = k \setminus \{0\}$. Then the equation

$$p(X, Y) = aZ^n ,$$

which we call a Thue equation, defines a regular curve C^a in $\boldsymbol{P}_k^2$, which we call a Thue curve. Let J^a be the Jacobian variety of C^a.

First assume that a and the coefficients of $p(X, Y)$ are in the ring $\mathfrak{o}_k$ of integers in k.

Let $d = [k : \boldsymbol{Q}]$ and $R_a = \operatorname{rank} J^a(k)$. Silverman [9] proved the following among others:

THEOREM 0.1 (Silverman [9]). *There is a constant $G = G(k, p(X, Y))$ such that if $a \in \mathfrak{o}_k \setminus \{0\}$ satisfies $|N_{\boldsymbol{Q}}^k a| > G$ and $|1 + \rho_n(a)| \leq 9/4$, then*

$$\#\{(x, y) \in \mathfrak{o}_k^2 \mid p(x, y) = a\} < n^{2n^2}(12n^3 d)^{R_a} ,$$

where $\rho_n(a)$ is a number which measures the defect in a of the n-th power freeness and differs from $e(a)$ in Theorem 0.2 below by addition of the multiple of $1/\log|N_{\boldsymbol{Q}}^k \mathfrak{a}|$ by a constant depending only on k and n.

He mapped $C^a(k)$ to $J^a(k)$ and estimated the number of lattice points which lie in a ball of $J^a(k) \otimes_{\boldsymbol{Z}} \boldsymbol{R}$.

On the other hand, Mumford [7] had asserted that the heights of rational points on the Jacobian which come from a curve under a certain map grow exponentially if the genus is greater than 1.

We here try to count the integral points by the technique of Silverman and the method of Mumford and to improve the result of Silverman. Consider the prime ideal decomposition of $a\mathfrak{o}_k$. Collecting the factors appropriately, we get a unique factorization of the form $a\mathfrak{o}_k = \mathfrak{a}\mathfrak{b}^n$, where $\mathfrak{a}$ is an integral ideal not divisible by the n-th power of any

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