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SHIMURA CURVES AS INTERSECTIONS OF HUMBERT SURFACES AND DEFINING EQUATIONS OF QM-CURVES OF GENUS TWO

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Abstract. Shimura curves classify isomorphism classes of abelian surfaces with quaternion multiplication. In this paper, we are concerned with a fibre space, the base space of which is a Shimura curve and fibres are curves of genus two whose jacobian varieties are abelian surfaces of the above type. We shall give an explicit defining equation for such a fibre space when the discriminant of the quaternion algebra is 6 or 10.

Introduction. Let A be a simple principally polarized abelian variety of dimension two over the complex number field $\boldsymbol{C}$, and $\mathrm{End}(A)$ the ring of endomorphisms of A. Then, as is well-known, the $\boldsymbol{Q}$-algebra $\mathrm{End}^{\circ}(A):=\mathrm{End}(A)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ is of one of the following types:

(i) a CM-field of degree four, (ii) an indefinite quaternion algebra,

(iii) a real quadratic field, or (iv) the rational number field $\boldsymbol{Q}$.

Let $\mathscr{A}_{2,1}$ be the moduli space of the isomorphism classes of abelian surfaces with principal polarization. The locus of each type in $\mathscr{A}_{2,1}$ has dimension 0, 1, 2, 3, respectively, whose irreducible components in the first three cases are called (i) CM-points, (ii) Shimura curves, and (iii) Humbert surfaces. On the other hand, it is also well-known that the Torelli map gives a birational morphism from $\mathscr{A}_{2,1}$ to the moduli space $\mathscr{M}_2$ of curves of genus two.

In this paper we are concerned with constructing, in a concrete way, an algebraic family of curves of genus two whose jacobian varieties belong to the case (ii) above. Namely, we wish to find out an equation for a fibre space, the base space of which is a Shimura curve and fibres are curves of genus two whose jacobian varieties have quaternion multiplications. Call such curves simply "QM-curves". We shall give defining equations over the rational number field $\boldsymbol{Q}$ for the algebraic family of QM-curves when the endomorphism ring is, generically, a maximal order $\mathcal{O}$ of the indefinite quaternion algebra $\boldsymbol{B}$ over $\boldsymbol{Q}$ which ramifies exactly at $\{2, 3\}$ or $\{2, 5\}$. To the best of our knowledge, not a single concrete example of *simple* QM-curves has been known before. Indeed, it is quite difficult to show that the jacobian variety of a given curve is simple.

The method of our construction is roughly as follows: In a classical work of Humbert [8], one can find general approach, as well as concrete solutions in some

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