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THE FUNCTOR OF A SMOOTH TORIC VARIETY

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Abstract. This paper describes the data needed to specify a map from a scheme to an arbitrary smooth toric variety. The description is in terms of a collection of line bundles and sections on the scheme which satisfy certain compatibility and nondegeneracy conditions. There is also a natural torus action on these collections. As an application, we show how homogeneous polynomials can be used to describe all maps from a projective space (or more generally a toric variety) to a smooth complete toric variety.

A map $Y \to \boldsymbol{P}_k^n$ is determined by a line bundle L on Y together with $n+1$ sections which do not vanish simultaneously. In fact, $\boldsymbol{P}_k^n$ is the variety representing the functor

(1) $\qquad Y \mapsto \{(L, u_0, \dots, u_n) : u_i \in H^0(Y, L) \text{ do not vanish simultaneously}\}/\sim\ ,$

where $\sim$ is the obvious equivalence relation. The goal of this paper is to generalize this description to the case of an arbitrary smooth toric variety.

We will work with schemes over a field k, and we will fix a smooth n-dimensional toric variety X determined by a fan Δ in $N_{\boldsymbol{R}} = \boldsymbol{R}^n$. As usual, M denotes the dual lattice of N and $\Delta(1)$ denotes the set of 1-dimensional cones of Δ. We will use $\sum_\rho$ to mean $\sum_{\rho \in \Delta(1)}$, and similarly for $\bigotimes_\rho$. Each $\rho \in \Delta(1)$ determines a divisor $D_\rho \subset X$ and a generator $n_\rho \in \rho \cap N$. Finally, let $\Delta_{\max}$ denote the set of maximal cones in Δ (i.e., those which are not proper faces of cones in Δ). Basic references for toric varieties are [3], [5] and [7].

1. Δ-collections and functors. If a fan Δ determines a smooth toric variety X, then we can generalize the data in (1) as follows:

DEFINITION 1.1. Given a scheme Y over k, a *Δ-collection on Y* consists of line bundles L_ρ and sections $u_\rho \in H^0(Y, L_\rho)$, indexed by $\rho \in \Delta(1)$, and isomorphisms $c_m : \bigotimes_\rho L_\rho^{\otimes \langle m, n_\rho \rangle} \simeq \mathcal{O}_Y$, indexed by $m \in M$, such that:

(i) (Compatibility) $c_m \otimes c_{m'} = c_{m+m'}$ for all $m, m' \in M$.

(ii) (Nondegeneracy) For each $y \in Y$, there is $\sigma \in \Delta_{\max}$ with $u_\rho(y) \neq 0$ for all $\rho \not\subset \sigma$.

A Δ-collection on Y is written (L_ρ, u_ρ, c_m). The compatibility condition on the isomorphisms c_m implies that $\sum_\rho [L_\rho] \otimes n_\rho = 0$ in $\mathrm{Pic}(Y) \otimes_{\boldsymbol{Z}} N$. However, the triviality of this sum is not sufficient: data of the Δ-collection includes an explicit choice of trivialization (the c_m's), which is not unique. The examples given below will show why

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