THE JOURNAL OF SYMBOLIC LOGIC
Volume 67, Number 3, Sept. 2002

$\aleph_0$-CATEGORICAL STRUCTURES WITH ARBITRARILY FAST GROWTH OF ALGEBRAIC CLOSURE

DAVID M. EVANS* AND M. E. PANTANO

§1. Introduction. Various results have been proved about growth rates of certain sequences of integers associated with infinite permutation groups. Most of these concern the number of orbits of the automorphism group of an $\aleph_0$-categorical structure $\mathscr{M}$ on the set of unordered n-subsets or on the set of n-tuples of elements of $\mathscr{M}$. (Recall that by the Ryll-Nardzewski Theorem, if $\mathscr{M}$ is countable and $\aleph_0$-categorical, the number of the orbits of its automorphism group $\mathrm{Aut}(\mathscr{M})$ on the set of n-tuples from $\mathscr{M}$ is finite and equals the number of complete n-types consistent with the theory of $\mathscr{M}$.) The book [Ca90] is a convenient reference for these results. One of the oldest (in the realms of 'folklore') is that for any sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers there is a countable $\aleph_0$-categorical structure $\mathscr{M}$ such that the number of orbits of $\mathrm{Aut}(\mathscr{M})$ on the set of n-tuples from $\mathscr{M}$ is greater than k_n for all n.

These investigations suggested the study of the growth rate of another sequence. Let $\mathscr{M}$ be an $\aleph_0$-categorical structure and X be a finite subset of $\mathscr{M}$. Let $\mathrm{acl}(X)$ be the algebraic closure of X, that is, the union of the finite X-definable subsets of $\mathscr{M}$. Equivalently, this is the union of the finite orbits on $\mathscr{M}$ of $\mathrm{Aut}(\mathscr{M})_{(X)}$, the pointwise stabiliser of X in $\mathrm{Aut}(\mathscr{M})$. Define

$$b_n = b_n(\mathscr{M}) = \max\{|\,\mathrm{acl}(X)|\colon X \subseteq \mathscr{M},\ |X| = n\}.$$

In [Ma86, Question 2], H. D. Macpherson posed the following problem:

PROBLEM. *Given any sequence $(a_n)_{n\in\mathbb{N}}$, is it always possible to find a transitive $\aleph_0$-categorical structure $\mathscr{M}$ such that $b_n \geq a_n$ for every n?*

For 'naturally occurring' examples of $\aleph_0$-categorical structures b_n does not grow faster than doubly exponentially, and in many cases algebraic closure is trivial (i.e., for every finite subset $X \subseteq \mathscr{M}$, $\mathrm{acl}(X) = X$).

Independently, D. M. Evans and H. D. Macpherson gave positive answers to the above question. The structures produced by Evans (see [Ev94], Theorem 4.3) had primitive and not 2-transitive automorphism groups and those of Macpherson had imprimitive automorphism groups. So the question remained as to whether $\mathscr{M}$ in Macpherson's problem could be multiply transitive. We show that it can be (although of course, in a $(k+1)$-transitive structure, the algebraic closure of k

Received June 11, 2001; revised December 21, 2001.
*Correspondence to first author.

0022-4812/02/6703-0001/$2.30