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ASYMPTOTIC CLASSES OF FINITE STRUCTURES

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§1. Introduction. In this paper we consider classes of finite structures where we have good control over the sizes of the definable sets. The motivating example is the class of finite fields: it was shown in [1] that for any formula $\phi(\bar{x}, \bar{y})$ in the language of rings, there are finitely many pairs $(d, \mu) \in \omega \times \mathbf{Q}^{>0}$ so that in any finite field $\boldsymbol{F}$ and for any $\bar{a} \in \mathbf{F}^m$ the size $|\phi(\mathbf{F}^n, \bar{a})|$ is "approximately" $\mu|\mathbf{F}|^d$. Essentially this is a generalisation of the classical Lang-Weil estimates from the category of varieties to that of the first-order-definable sets.

Motivated by this, we say that finite fields form a 1-*dimensional asymptotic class*. Macpherson and Steinhorn in [5] have studied these classes in abstract. Generalising this, in 2.1 below we define N-*dimensional asymptotic classes* for natural numbers $N \geq 1$, and begin to develop their general theory. In that definition, we have relaxed the asymptotic conditions (the meaning of "approximately" above), to encompass to the widest possible range of examples. We prove in corollary 2.8 that our classes lie within the general context of supersimple theories of finite rank.

In section 3 we consider how to define and interpret asymptotic classes inside one another, and in proposition 3.7 we show that the property of being an asymptotic class is invariant under bi-interpretations. In section 4 we give some examples of asymptotic classes, in particular, in proposition 4.1 we show that the smoothly approximable structures comprehensively studied in [2] fit into our framework. In section 5 we re-examine the relationship between dimension and D-rank. In section 6 we consider stable asymptotic classes. We show that stability can be detected within the finite structures in our context, and in proposition 6.5, we observe that stable asymptotic classes are locally modular.

NOTATION. If $\mathscr{U}$ is a (non-principal) ultrafilter on a set I and $\{M_i : I\}$ is a collection of $\mathscr{L}$-structures, we denote the ultraproduct by $P = \prod_{i \in I} M_i/\mathscr{U}$, and for $\bar{a} = (a_1, \dots, a_n) \in P^n$, we shall write $\bar{a}(M) = (a_1(M), \dots a_n(M))$ to mean the tuple of co-ordinates of $\bar{a}$ in M, so that for each $j \in \{1, \dots, n\}$ we have $a_j = \prod_{i \in I} a_j(M_i)/\mathscr{U}$.

If M is an $\mathscr{L}$-structure, we write $\mathrm{Def}(M)$ for the collection of all parameter-definable sets in all powers of M.

§2. Basic definitions and lemmas.

2.1. DEFINITION. *Let $\mathscr{L}$ be a countable first order language, $N \in \omega$, and $\mathscr{C}$ a class of finite $\mathscr{L}$-structures. Then we say that $\mathscr{C}$ is a an N*-**dimensional asymptotic**

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