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# On the compact convex base of a dual cone

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Let $X$ be a locally convex Hausdorff linear topological space over $R$, where $R$ is the field of real numbers endowed with its usual topology, $X^{*}$ be its topological dual and $K$ be a closed proper cone with vertex $\theta$, i. e., a closed subset of $X$ with the following properties: i) $K+K\subset K$, ii) $\lambda K\subset K$ for all $\lambda\geqq 0$, and iii) $K\cap(-K)=\{\theta\}$, where $\theta$ denotes the zero element of the linear space $X$. Then $K$ allows us to introduce, by virtue of "$x\leqq y$ if $y-x\in K$", a partial order $\leqq$, under which $X$ is an ordered linear space with possitive cone $K$. Let $\Delta$ be a non-empty subset of the dual cone $K^{*}=\{x^{*}: x^{*}\in X^{*}. x^{*}(x)\geqq 0$ for all $x\in K.\}$ satisfying the following conditions:

(1) if $x^{*}(x)\geqq 0$ for all $x^{*}\in\Delta$, then $x\in K$;

(2) $\Delta$ is strongly compact and convex;

(3) $\theta^{*}\notin\Delta$.

Here $\theta^{*}$ is reserved for the zero element of $X^{*}$.

The use of $\Delta$ as a set of price systems is justified, when it is intended to treat the infinite-dimensional commodity space (see [2]). Although, in the finite-dimensional case, an example of $\Delta$ is easily found, it is not always easy and even impossible to find such an example in the infinite-dimensional case. The purpose of this paper is to discuss the existence of $\Delta$ in the infinite-dimensional case. In the infinite-dimensional Banach space, there exists no non-empty subset $\Delta$ of $K^{*}$ with $X^{*}=K^{*}-K^{*}$ satisfying (1), (2) and (3). In fact, the following theorem holds:

THEOREM. *Let $X$ be a Banach space, $K$ be a closed proper cone in $X$ and $K^{*}$ be its dual cone. Assume $X^{*}=K^{*}-K^{*}$. If there exists a non-empty subset $\Delta$ of $K^{*}$ which satisfies the above conditions* (1), (2) *and* (3), *then $X$ is finite-dimensional.*

PROOF. Let $(\bigcup_{\lambda\geqq 0}\lambda\Delta)^{w-}$ denote the weak*-closure of $\bigcup_{\lambda\geqq 0}\lambda\Delta$. Then obviously $K^{*}\supset(\bigcup_{\lambda\geqq 0}\lambda\Delta)^{w-}$.

Suppose that $x_{0}^{*}\notin(\bigcup_{\lambda\geqq 0}\lambda\Delta)^{w-}$ for some $x_{0}^{*}\in K^{*}$. Then, by making use of the separation theorem, there exists an $x_{0}\in X$ such that

$$\inf_{x^{*}\in\bigcup_{\lambda\geqq 0}\lambda\Delta} x^{*}(x_{0})\geqq 0>x_{0}^{*}(x_{0}) .$$