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Classification of exotic circles of $PL_{+}(S^{1})$

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Abstract. Let G be a subgroup of the group $\mathrm{Homeo}_{+}(S^{1})$ of orientation preserving homeomorphisms of the circle. An exotic circle of G is a subgroup of G which is topologically conjugate to $SO(2)$ but not conjugate to $SO(2)$ in G. The existence of an exotic circle shows us the fact that the subgroup G is far from being a Lie group. We previously proved that the group $PL_{+}(S^{1})$ of orientation preserving piecewise linear homeomorphisms of the circle has exotic circles. We give a more explicit construction of exotic circles of $PL_{+}(S^{1})$ and classify all exotic circles up to PL conjugacy.

Key words: topological circle, exotic circle, $PL_{+}(S^{1})$, topologically conjugate, PL conjugate, total derivative, half total derivative.

Introduction

Let G be a Lie group and M an oriented manifold of class C^{k} $(1\leq k\leq\infty)$. Let $\mathrm{Diff}_{+}^{k}(M)$ denote the group of all C^{k} diffeomorphisms of M. A topological action is a continuous map $\varphi : G\times M\rightarrow M$ such that

1) $\varphi_{e}(x)=x$,
2) $\varphi_{gh}(x)=\varphi_{g}(\varphi_{h}(x))$.

where e is the unit of G and $\varphi_{g}(x)=\varphi(g, x)$. D. Montgomery and L. Zippin proved the following theorem ([4]).

Theorem 0.1 *Let φ be a topological action. If every φ_{g} belongs to $\mathrm{Diff}_{+}^{k}(M)$ then φ is a map of class C^{k}.*

In the case where $G=M=S^{1}$, this theorem implies the following corollary.

Corollary 0.2 *If every $h\circ R_{x}\circ h^{-1}$ is contained in $\mathrm{Diff}_{+}^{k}(S^{1})$, then h belongs to $\mathrm{Diff}_{+}^{k}(S^{1})$. Here, $R_{x} : S^{1}\rightarrow S^{1}$ is the rotation of S^{1} by x, i.e., $R_{x}(y)=x+y$.*

Indeed, for $\varphi(x, y)=h\circ R_{x}\circ h^{-1}(y)$. $\varphi : S^{1}\times S^{1}\rightarrow S^{1}$ is a topological action with $\varphi_{x}\in \mathrm{Diff}_{+}^{k}(S^{1})$. Then φ is of class C^{k} by Theorem 0.1. Fix

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