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Estimates of spherical derivative of meromorphic functions

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Abstract. The spherical derivative $f^{\#}=|f'|/(1+|f|^{2})$ of f meromorphic in $D=\{|z|<1\}$ is estimated from above and below in terms of various geometrical quantities, for example, $\delta^{\#}(z, f)$, $\rho(z, f)$, and $\rho_{au}(z, f)$, in several theorems. A necessary and sufficient condition for $(1-|z|^{2})f^{\#}(z)$ to be bounded in D is that there exists r, $0<r\leq 1$, such that $f(w)\neq-1/\overline{f(z)}$ for all z, $w\in D$ satisfying $|w-z|/|1-\overline{z}w|<r$. Also, $(1-|z|^{2})f^{\#}(z)$ is bounded in D if and only if $\delta^{\#}(z, f)/\rho_{au}(z, f)$ is bounded in D minus the points z where $f^{\#}(z)=0$. Applications to evaluating the Poincaré density in a plane domain will be considered.

Key words: normal meromorphic function; antipodal point; spherical and Poincaré distances; spherical derivative of meromorphic function; Poincaré density; Bloch function.

1. Introduction

Let a function f be meromorphic in the disk $D=\{|z|<1\}$. The spherical derivative $f^{\#}(z)$ of f at $z\in D$ is defined by $f^{\#}(z)=|f'(z)|/(1+|f(z)|^{2})$, if $f(z)\neq\infty$, and $f^{\#}(z)=|(1/f)'(z)|$, if $f(z)=\infty$. Then $f^{\#}=(1/f)^{\#}$ in D, where the constant function ∞ is regarded as a meromorphic function, so that $\infty^{\#}=0$. One can prove that $f^{\#}$ is continuous in D. Actually we shall be mainly concerned with a kind of derivative of f, namely,

$$\Phi_{f}(z)=(1-|z|^{2})f^{\#}(z), \quad z\in D.$$

We call f normal if Φ_{f} is bounded in D; see [LV] for the details. Let $\rho_{a}(z, f)$ be the maximum of r, $0<r\leq 1$, such that $f(w)\neq-1/\overline{f(z)}$, the antipodal point of $f(z)$, for all w in the Apollonius disk, or the non-Euclidean disk

$$\Delta(z, r)=\left\{w;\left|\frac{w-z}{1-\overline{z}w}\right|<r\right\}$$

of center z and the non-Euclidean radius $\operatorname{arctanh} r$. Such a $\rho_{a}(z, f)>0$

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