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The Luzin Theorem for Higher-Order Derivatives

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1. Introduction

In 1917, Luzin ([2]; see also [4]) proved a surprising result: For any Lebesgue measurable function $f : \mathbb{R} \to \mathbb{R}$ there is a continuous a.e. differentiable function g such that $g' = f$ almost everywhere. This is surprising even for the function $f(x) = 1/x$ because the antiderivative of f is discontinuous and, in fact, unbounded at 0. In this case, we correct the antiderivative by adding continuous functions that are differentiable almost everywhere with derivative equal to zero but that are not constant (one example of such a function is a Cantor staircase).

The original proof due to Luzin is purely one-dimensional and offers no guidance toward a proof in higher dimensions. However, in 2008 Moonens and Pfeffer [3] proved the following generalization:

> *Let U be an open subset of $\mathbb{R}^N$. Given any Lebesgue measurable function $f : U \to \mathbb{R}^N$, there is an a.e. differentiable function $g \in C(\mathbb{R}^N)$ such that $\nabla g = f$ almost everywhere.*

The goal of this paper is to extend the results to include higher-order derivatives.

For an m-times differentiable function g defined in an open subset $U \subset \mathbb{R}^N$, we write

$$D^m g = (D^\alpha g)_{|\alpha|=m}$$

to denote the collection of all partial derivatives of order m. Our main result reads as follows.

Theorem 1.1. *Let $f = (f^\alpha)_{|\alpha|=m}$ be a Lebesgue measurable function defined in an open set $U \subset \mathbb{R}^n$. Then there is a function $g \in C^{m-1}(\mathbb{R}^n)$ that is m-times differentiable a.e. and such that*

$$D^m g = f \quad \text{a.e. in } U;$$

that is,

$$D^\alpha g = f^\alpha \quad \text{a.e. in } U \quad \text{for } |\alpha| = m.$$

Moreover, for any $\sigma > 0$, the function g may be chosen such that

$$\|D^\gamma g\|_\infty < \sigma \quad \text{for every } |\gamma| < m.$$

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