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A Strong Comparison Principle for Plurisubharmonic Functions with Finite Pluricomplex Energy

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1. Introduction

Let Ω be a hyperconvex domain in $\mathbb{C}^n$—in other words, Ω is a bounded, open, and connected subset of $\mathbb{C}^n$ and there exists a continuous plurisubharmonic (psh) function ρ in Ω such that $\{z \in \Omega : \rho(z) < -c\} \subset\subset \Omega$ for any constant $c > 0$. Denote by $\mathrm{PSH}(\Omega)$ the set of psh functions in Ω and by $\mathrm{PSH}^-(\Omega)$ the set of nonpositive psh functions in Ω. Write $d = \partial + \bar\partial$ and $d^c = i(\bar\partial - \partial)$. The complex Monge–Ampère operator $(dd^c)^n$ is well-defined on all locally bounded psh functions; see Bedford and Taylor's fundamental paper [BeT]. It plays a central role in pluripotential theory just as the Laplace operator does in classical potential theory. We refer to excellent surveys [Be; Ki2] and books [Kl; K] for references. The monotone convergence theorem and the comparison principle of Bedford and Taylor are both of theoretical interest and extremely useful in pluripotential theory. They are used in almost all papers dealing with the Monge–Ampère operator. We know that the comparison principle not only gives the uniqueness theorem of the Dirichlet problem for the Monge–Ampère operator but also is one of main tools in solving Monge–Ampère equations. In [X1] we obtained the following type of comparison theorem.

Strong Comparison Principle. *Let $u, v \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ be such that* $\liminf_{z\to\partial\Omega}(u(z) - v(z)) \ge 0$. *Then for any constant $r \ge 0$ and all $w_j \in \mathrm{PSH}(\Omega)$ with $-1 \le w_j \le 0$, $j = 1, 2, \dots, n$, we have*

$$\frac{1}{(n!)^2}\int_{u<v}(v-u)^n\, dd^c w_1 \wedge \dots \wedge dd^c w_n + \int_{u<v}(r - w_1)(dd^c v)^n \le \int_{u<v}(r - w_1)(dd^c u)^n.$$

The strong comparison principle has many applications (see [X1; X2; X3]) and, moreover, it implies several important inequalities in pluripotential theory. Let's show some of its direct consequences.

Proposition 1 (First version of Bedford and Taylor's comparison principle; see [BeT]). *If $u, v \in \mathrm{PSH} \cap L^\infty(\Omega)$ satisfy* $\liminf_{z\to\partial\Omega}(u(z) - v(z)) \ge 0$, *then*

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