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Sharp Bounds for Eigenvalues of Triangles

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1. Introduction

The purpose of this paper is to prove the following theorem.

Theorem 1.1. *Let T be a triangle in a plane and suppose T has area A and perimeter L. Then the first eigenvalue λ_T of the Dirichlet Laplacian on T satisfies*

$$\frac{\pi^2 L^2}{16A^2} < \lambda_T \leq \frac{\pi^2 L^2}{9A^2}. \tag{1.1}$$

The constants 9 and 16 are optimal, and equality in the upper bound holds only for the equilateral triangle.

The lower bound was proved in a more general context in [3]. In Section 6 we show that for "tall" isosceles triangles there is an asymptotic equality in the lower bound. Hence it is impossible to decrease the constant 16.

The upper bound was recently stated as a conjecture in [2], and numerical evidence for its validity is given in [1]. Bounds of this form but with different constants have been the subject of many papers in the literature. The eigenvalue of any doubly connected domain is bounded above by the same fraction but with the constant 4; see [7] and remarks in [5]. There is also a sharper upper bound due to Freitas [2] that is not of this form, but it seems that in the worst case ("tall" isosceles triangle) it gives the constant 6 and in the best (equilateral) 9. Observe that the constant 9 cannot be improved because equilateral triangles give equality in the upper bound of Theorem 1.1.

It is worth mentioning that the same theorem can be equivalently stated in terms of the inradius $R = 2A/L$ of the triangle.

Theorem 1.2. *Let T be a triangle in a plane and let T have inradius R. Then the first eigenvalue λ_T of the Dirichlet Laplacian on T satisfies*

$$\frac{\pi^2}{4} < \lambda_T R^2 \leq \frac{4\pi^2}{9}. \tag{1.2}$$

The equality in the upper bound holds only for the equilateral triangle.

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