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On Weyl Sums over Primes and Almost Primes

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1. Introduction

In this paper we pursue estimates for the exponential sum

$$f(\alpha) = \sum_{P \le p < 2P} e(\alpha p^k), \tag{1.1}$$

where α is a real number, k is a positive integer, $e(z) = \exp\{2\pi i z\}$, and the summation is over prime numbers. This sum was introduced as a tool in analytic number theory by Vinogradov in the late 1930s. In 1937, Vinogradov developed an ingenious new method for estimating sums over primes and applied that method to obtain the first unconditional estimate for $f(\alpha)$ with $k = 1$. That estimate is the main novelty in his celebrated proof [25] that every sufficiently large odd integer is the sum of three primes. In the sharper form given in [27, Chap. 6], Vinogradov's result states (essentially) that, if a and q are integers satisfying

$$q \ge 1, \quad (a,q) = 1, \quad |q\alpha - a| < q^{-1}, \tag{1.2}$$

then

$$f(\alpha) \ll q^{\varepsilon} P (q^{-1} + P^{-2/5} + qP^{-1})^{1/2} \tag{1.3}$$

for any fixed $\varepsilon > 0$. Vinogradov also obtained estimates for $f(\alpha)$ with $k \ge 2$ and used them to give the first unconditional results concerning the Waring–Goldbach problem. When $k \ge 2$, the sharpest estimates for $f(\alpha)$ obtained by Vinogradov's method were proven by Harman [3; 4]. In particular, he showed in [3] that if (1.2) holds then

$$f(\alpha) \ll P^{1+\varepsilon} (q^{-1} + P^{-1/2} + qP^{-k})^{4^{1-k}}. \tag{1.4}$$

Vinogradov's approach does not rely heavily on the particular form of the phases in (1.1) and can be applied to more general sums (see [3; 28]). In 1991, Baker and Harman [1] demonstrated that, using the Diophantine properties of the sequence am^k/q, one can derive sharper bounds for $f(a/q)$ with $k \ge 2$. They proved (essentially) that if q is near $P^{k/2}$ and $(a,q) = 1$ then

$$f(a/q) \ll P^{1-\rho(k)+\varepsilon},$$

where $\rho(2) = \frac{1}{7}$ and $\rho(k) = \frac{2}{3} \times 2^{-k}$ for $k \ge 3$. They applied this bound to obtain new results on the distribution of αp^k modulo 1. On the other hand, research on

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