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On Representation of Integers by Sums of a Cube and Three Cubes of Primes

Xiumin Ren & Kai-Man Tsang

1. Introduction

We consider the expression of positive integers n as the sum of a cube and three cubes of primes; that is,

$$n = m^3 + p_2^3 + p_3^3 + p_4^3, \tag{1.1}$$

where m is a positive integer and p_j are primes. In 1949, Roth [8] proved that almost all positive integers n can be written as (1.1). More precisely, let $E(N)$ denote the number of positive integers $n \le N$ that cannot be written in the form (1.1); then Roth's theorem actually stated that $E(N) \ll N \log^{-A} N$ for arbitrary $A > 0$. In 1995, Brüdern [1] proved that the same exceptional set estimate holds for the number of positive integers $n \equiv 4 \pmod{18}$, not exceeding N, that cannot be written in the form (1.1) with m restricted to a P_4-number. Later Kawada [2] further strengthened this by replacing m by a P_3-number. All these results can be viewed as approximations to the conjecture that all sufficiently large integers satisfying some necessary congruence conditions are the sum of four cubes of primes. As is well known, the quality of the approximation is indicated in the upper bound of $E(N)$. Roth's theorem has been improved by Ren [5] to $E(N) \ll N^{169/170}$ and by Ren and Tsang [7] to $E(N) \ll N^{1271/1296+\varepsilon}$. These improvements were obtained via new approaches for enlarging major arcs in the circle method used (see e.g. [4; 5; 7]). In this paper, based on the major arcs estimate in [7], we use some new ideas to handle the minor arcs and prove the following.

Theorem 1. *For $E(N)$ as just defined, we have*

$$E(N) \ll N^{17/18+\varepsilon}.$$

Notation. As usual, $\Lambda(n)$ stands for the von Mangoldt function. In our statement, N is a large positive integer and $L = \log N$. The notation $r \sim R$ means $R < r \le 2R$. The letters ε and A denote positive constants that are (respectively) arbitrarily small and arbitrarily large; they may assume different values at each occurrence.

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