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Borel Images and Analytic Functions

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1. Introduction

The σ-algebra $\mathcal{B}$ of Borel sets in $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is, by definition, the smallest σ-algebra that contains all open sets.

It is well known that the preimage of a Borel set under a continuous mapping is again a Borel set, whereas the image of a Borel set need not be a Borel set. See [13; 25; 26] for a discussion from the point of view of descriptive set theory.

We say that the mapping f *preserves Borel sets on* A if f is defined on A and if

$$B \subset A,\ B \in \mathcal{B} \implies f(B) \in \mathcal{B}.$$

There seems to be no standard name for this property.

The notion of injectivity plays an important role. Lusin and Suslin showed that any injective Borel measurable map $f : B \to \mathbb{C}$ $(B \in \mathcal{B})$ preserves Borel sets. In [7] it was shown that conformal maps of $\mathbb{D}$ into $\hat{\mathbb{C}}$ preserve Borel sets for radial limits.

Lusin and Purves have characterized the functions that preserve Borel sets in terms of the number of preimages at the points in their domain. All that we shall prove is based on the following result (see [14, p. 406; 21]).

Theorem (Lusin–Purves). *Let $A \in \mathcal{B}$ and let $f : A \to \hat{\mathbb{C}}$ have the property that $f^{-1}(E) \in \mathcal{B}$ for every $E \in \mathcal{B}$. Then f preserves Borel sets on A if and only if the set*

$$\{w : w = f(z) \text{ for uncountably many } z \in A\}$$

is countable.

We shall be interested in functions that are analytic (holomorphic) in the unit disk $\mathbb{D}$ and are continuous in $\bar{\mathbb{D}}$ (see Section 2) or have radial limits on subsets of $\mathbb{T} = \partial\mathbb{D}$ (see Sections 3 and 4). In Section 5 we characterize the plane domains whose

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