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The Equivariant Cohomology Ring of Regular Varieties

Michel Brion & James B. Carrell

0. Introduction

Let $\mathfrak{B}$ denote the upper triangular subgroup of $\mathrm{SL}_2(\mathbb{C})$, $\mathfrak{T}$ its diagonal torus and $\mathfrak{U}$ its unipotent radical. A complex projective variety Y is called *regular* if it is endowed with an algebraic action of $\mathfrak{B}$ such that the fixed-point set $Y^{\mathfrak{U}}$ is a single point. Associated to any regular $\mathfrak{B}$-variety Y is a remarkable affine curve $\mathcal{Z}_Y$ with a $\mathfrak{T}$-action, which was studied in [7]. In this note, we show that the coordinate ring $\mathbb{C}[\mathcal{Z}_Y]$ is isomorphic with the equivariant cohomology ring $H^*_{\mathfrak{T}}(Y)$ with complex coefficients when Y is smooth or, more generally, is a $\mathfrak{B}$-stable subvariety of a regular smooth $\mathfrak{B}$-variety X such that the restriction map from $H^*(X)$ to $H^*(Y)$ is surjective. This isomorphism is obtained as a refinement of the localization theorem in equivariant cohomology; it applies for example to Schubert varieties in flag varieties and to the Peterson variety studied in [11]. Another application of our isomorphism is a natural algebraic formula for the equivariant push-forward.

1. Preliminaries

Let $\mathfrak{B}$ be the group of upper triangular 2×2 complex matrices of determinant 1. Let $\mathfrak{T}$ (resp. $\mathfrak{U}$) be the subgroup of $\mathfrak{B}$ consisting of diagonal (resp. unipotent) matrices. We have isomorphisms $\lambda \colon \mathbb{C}^* \to \mathfrak{T}$ and $\varphi \colon \mathbb{C} \to \mathfrak{U}$, where

$$\lambda(t) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \text{and} \quad \varphi(u) = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix},$$

which together satisfy the relation

$$\lambda(t)\varphi(u)\lambda(t^{-1}) = \varphi(t^2 u). \tag{1}$$

Consider the generators

$$\mathcal{V} = \dot{\varphi}(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{W} = \dot{\lambda}(1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

of the Lie algebras $\mathrm{Lie}(\mathfrak{U})$ and $\mathrm{Lie}(\mathfrak{T})$, respectively. Then $[\mathcal{W}, \mathcal{V}] = 2\mathcal{V}$, and

$$\mathrm{Ad}(\varphi(u))\mathcal{W} = \mathcal{W} - 2u\mathcal{V} \tag{2}$$

for all $u \in \mathbb{C}$.

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