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Sharp Sobolev Inequalities in Critical Dimensions

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1. Introduction

Let $K \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N$ $(N \geq 2K+1)$ be a regular bounded domain in $\mathbb{R}^N$. We consider the semilinear polyharmonic problem

$$(-\Delta)^K u = \lambda u + |u|^{s-2}u \ \text{ in } \Omega, \tag{1}$$

where

$$s := \frac{2N}{N-2K}$$

denotes the critical Sobolev exponent. For $K = 1$, Brezis and Nirenberg [3] studied the existence of positive solutions of (1) with homogenous Dirichlet boundary conditions

$$u = 0 \ \text{ on } \partial\Omega. \tag{2}$$

They discovered the following remarkable phenomenon: the qualitative behavior of the set of solutions of (1) and (2) is highly sensitive to N, the dimension of the space. To state their result precisely, let us denote by $\lambda_1 > 0$ the first eigenvalue of $-\Delta$ in Ω. Brezis and Nirenberg showed for $K = 1$ that: (a) in dimension $N \geq 4$, there exists a positive solution of (1) and (2) if and only if $\lambda \in (0, \lambda_1)$; while (b) in dimension $N = 3$ and when $\Omega = B_1$ is the unit ball, there exists a positive solution of (1) and (2) if and only if $\lambda \in (\lambda_1/4, \lambda_1)$.

Pucci and Serrin [13] later considered the general polyharmonic problem (1) with $K \geq 1$ and with homogenous Dirichlet boundary conditions given by

$$D^k u = 0 \ \text{ on } \partial\Omega \quad \text{for } \ k = 0, \dots, K-1. \tag{3}$$

Here $D^k u$ denotes any derivative of order k of the function u. Pucci and Serrin were interested in the existence of nontrivial radial solutions of (1) subject to the boundary conditions (3) in the case $\Omega = B_1$. They introduced the notion of *critical dimensions* for (1) and (3) as the dimensions N for which radial solutions exist only for $\lambda > \lambda^*$, where $\lambda^* > 0$. Moreover, they conjectured that, given $K \geq 1$, the critical dimensions are given by $2K+1 \leq N \leq 4K-1$. It is shown in [7] that the dimensions $N \geq 4K$ are not critical, and the conjecture of Pucci and Serrin has been partially solved; see [1; 4; 8; 13; 14] and the references therein.

The critical dimensions are intimately related to the existence of sharp Sobolev inequalities. Indeed, motivated by the nonexistence results in [3], Brezis and Lieb

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