ON LOCAL BALANCE AND N-BALANCE IN SIGNED GRAPHS

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A *signed graph* or *s-graph* [2] is obtained from a linear graph when some of its lines are regarded as positive and the remaining lines as negative. The *sign of a cycle* is the product of the signs of its lines. An s-graph is *balanced* if all its cycles are positive. Two characterizations of balanced s-graphs were given in [2], Theorems 2 and 3. The definitions of all terms used here may be found in [2].

For certain applications of the theory of signed graphs to problems in social psychology, one is interested only in the cycles through a designated point. For other psychological considerations, one considers only cycles of length not exceeding N. These viewpoints lead to the definitions of local balance and N-balance in s-graphs. Some properties of these kinds of balance will be derived in this note. A detailed discussion of the relevance of the notion of balance of s-graphs to psychological theory is given in [1].

An s-graph G is *locally balanced at the point* P, or briefly, G is *balanced at* P, if all cycles containing P are positive. Theorem 1 below shows the interdependence of local balance and articulation points. An *articulation point* of a connected graph is a point whose removal results in a disconnected graph. We first require an extension of the sign of a path or cycle to any set of lines of G. Let L_1 be a subset of L, the set of all lines of G. The *sign* of L_1 is the product of the signs of the lines of L_1. The previous definitions of the sign of a path or a cycle are of course specializations of this one. If L_1, L_2 are subsets of L, then $L_1 \oplus L_2$ denotes the symmetric difference, or set union modulo 2, of L_1 and L_2. Let $s(L_1)$ denote the sign of L_1. It is convenient to prove two lemmas before taking up the theorem on local balance.

LEMMA 1. $s(L_1 \oplus L_2 \oplus \cdots \oplus L_n) = s(L_1) \cdot s(L_2) \cdot \cdots \cdot s(L_n)$.

Proof. For $n = 1$, the lemma is trivial. When $n = 2$, we make use of the usual formula $L_1 + L_2 = (L_1 - L_2) \cup (L_2 - L_1)$, which expresses $L_1 \oplus L_2$ as a union of disjoint sets. By definition of the sign of L_1, we have $s(L_1) = \Pi_{\lambda \in L_1} s(\lambda)$. Now L_1 can be expressed as the union of two disjoint sets:

$$L_1 = (L_1 - L_2) \cup (L_1 \cap L_2).$$

Thus

$$s(L_1) = s(L_1 - L_2) \cdot s(L_1 \cap L_2) \quad \text{and} \quad s(L_2) = s(L_2 - L_1) \cdot s(L_1 \cap L_2).$$

Hence

$$\begin{aligned} s(L_1) \cdot s(L_2) &= s(L_1 - L_2) \cdot s(L_2 - L_1) \cdot (s(L_1 \cap L_2))^2 \\ &= s(L_1 - L_2) \cdot s(L_2 - L_1) \\ &= s(L_1 \oplus L_2). \end{aligned}$$

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