Michigan Math. J. 47 (2000)

C^α-Compactness and the Calabi Flow on Kähler Surfaces with Negative Scalar Curvature

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1. Introduction

Let M be a compact Kähler m-manifold that has a Kähler metric $ds^2 = g_{\alpha\bar\beta}dz^\alpha \otimes d\bar z^\beta$. Then it is known that, for the Ricci curvature tensor $R_{\alpha\bar\beta} = -(\partial^2/\partial z^\alpha\partial\bar z^\beta)\log\det(g_{\lambda\bar\mu})$, $\frac{\sqrt{-1}}{2\pi}R_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta$ is a closed $(1,1)$-form and its cohomology class is equal to the first Chern class $C_1(M)$. Conversely, it was Calabi who asked if, for any closed $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\tilde R_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta$ that is cohomologous to $C_1(M)$, can one find a Kähler metric $\tilde g_{\alpha\bar\beta}$ on M such that $\tilde R_{\alpha\bar\beta}$ is the Ricci curvature tensor of $\tilde g_{\alpha\bar\beta}$? As a consequence of Aubin and Yau's results, one can find a Kähler–Einstein metric on M with $C_1(M) = 0$ or $C_1(M) < 0$. When $C_1(M) > 0$, the space of Kähler–Einstein metrics are invariant under automorphism group. However, the existence does not always hold in general [F; M; T; TY].

Instead of the Kähler–Einstein metric, we consider the notion of extremal metrics due to Calabi [C1]. Namely, fix a Kähler class $\Omega_0 = [\omega_0]$ on a compact Kähler manifold M and denote by H_{Ω_0} the space of all Kähler metrics with the same fixed Kähler class Ω_0. Now consider the functional $\Phi\colon H_{\Omega_0}\to\mathbf{R}$,

$$\Phi(g) = \int_M R^2\,d\mu_g,$$

where R denotes the scalar curvature of g. A critical point of Φ is called an *extremal metric*. In particular, any Kähler–Einstein metric is an extremal metric that also minimizes $\int_M R^2\,d\mu_g$ in H_{Ω_0}. Furthermore, if $\Omega_0 = C_1(M) > 0$ and if there exist no nonzero holomorphic vector fields on M, then an extremal metric is a Kähler–Einstein metric. On the other hand, there exist some obstructions to the existence of extremal metrics due to Calabi [C2], LeBrun [L], Levine [Le], and Burns and deBartolomeis [BB]. However, so far there is no known example of a compact Kähler manifold M with $C_1(M) > 0$ and no nonzero holomorphic tangent vector field that does not carry any extremal metric. Concerning the existence of extremal metrics, Calabi has asked whether one can always minimize the Φ in H_{Ω_0} on M if there exist no nonzero holomorphic tangent vector fields and if the tangent bundle of M is stable (see [C1; D; SY; UY]).

Throughout this note, we consider a compact Kähler surface M (see Remark 2.2) with a fixed Kähler class $\Omega_0 = [\omega_0]$ for $\omega_0 = \frac{\sqrt{-1}}{2\pi}\overset{0}{g}_{\alpha\bar\beta}dz^\alpha\wedge d\bar z^\beta$. For any metric

Received November 23, 1998. Revision received May 23, 2000.
Research supported in part by NSC and CTS Fellowship.