Classification Theorem for a Class of Flat Connections and Representations of Kähler Groups

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1. Introduction

1.1

Let M be a compact Kähler manifold. For a matrix Lie group G, the representation variety $\mathcal{M}_G$ of the fundamental group $\pi_1(M)$ is defined as the quotient $\mathrm{Hom}(\pi_1(M), G)//G$. Here G acts on the set $\mathrm{Hom}(\pi_1(M), G)$ by pointwise conjugation: $(gf)(s) = gf(s)g^{-1}$, $s \in \pi_1(M)$. A study of geometric properties of $\mathcal{M}_G$ is of interest because of the relation to the problem of classifying Kähler groups (a problem posed by J.-P. Serre in the 1950s). For a simply connected nilpotent Lie group G, every element of $\mathcal{M}_G$ is uniquely determined by a d-harmonic nilpotent matrix 1-form ω on M such that $\omega \wedge \omega$ represents 0 in the corresponding de Rham cohomology group. This follows, for example, from a theorem on formality of a compact Kähler manifold [DGMS]. The main result of our paper gives, in particular, a similar description for elements of $\mathcal{M}_G$ with a simply connected solvable Lie group G. Our arguments are straightforward and based on cohomology techniques only. As a consequence of the main theorem we obtain several results on the structure of Kähler groups. We now proceed to a formulation of the results.

It is well known that $\mathcal{M}_{\mathrm{GL}_n(\mathbb{C})}$ is equivalently characterized as moduli spaces of flat bundles over M with structure group $\mathrm{GL}_n(\mathbb{C})$. In this paper we consider a family of C^∞-trivial complex flat vector bundles over M. Every bundle from this family is determined by a flat connection on the trivial bundle $M \times \mathbb{C}^n$, that is, by a matrix-valued 1-form ω on M satisfying

$$d\omega - \omega \wedge \omega = 0. \tag{1.1}$$

Moreover, we assume that the $(0,1)$-component ω_2 of ω is an upper triangular matrix form. Denote this class of connections by $\mathcal{A}_n^t$.

Remark 1.1. Connections from $\mathcal{A}_n^t$ determine (by iterated path integration) all representations of $\pi_1(M)$ into simply connected complex solvable Lie groups.

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