The Composition Operators on the Space of Dirichlet Series with Square Summable Coefficients

Julia Gordon & Håkan Hedenmalm

0. Introduction

Let $\mathcal{H}$ be the space of Dirichlet series with square summable coefficients; $f \in \mathcal{H}$ means that the function has the form

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \tag{0.1}$$

with $\sum_{n=1}^{\infty} |a_n|^2 < +\infty$. By the Cauchy–Schwarz inequality, the functions in $\mathcal{H}$ are all holomorphic on the half-plane $\mathbb{C}_{1/2} = \{ s \in \mathbb{C} : \Re s > \frac{1}{2} \}$. The coefficients $\{a_n\}_n$ can be retrieved from the holomorphic function $f(s)$, so that $\|f\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |a_n|^2$ defines a Hilbert space norm on $\mathcal{H}$. We consider the following problem.

For which analytic mappings $\Phi\colon \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ is the composition operator $\mathcal{C}_\Phi(f) = f \circ \Phi$ a bounded linear operator on $\mathcal{H}$?

In this paper, a complete answer to this question is found. In the process, we encounter the space $\mathcal{D}$ of functions f, which in some (possibly remote) half-plane $\mathbb{C}_\theta = \{ s \in \mathbb{C} : \Re s > \theta \}$ ($\theta \in \mathbb{R}$) admit representation by a convergent Dirichlet series (0.1). It is, in a sense, a space of germs of holomorphic functions. It is important to note that if a Dirichlet series converges on $\mathbb{C}_\theta$ then it converges absolutely and uniformly on $\mathbb{C}_\vartheta$, provided $\vartheta > \theta + 1$ (see e.g. [3]). In terms of the coefficients, $f \in \mathcal{D}$ means that a_n grows at most polynomially in the index variable n. We shall use the notation $\mathbb{C}_+$ to denote the right half-plane, $\mathbb{C}_+ = \{ s \in \mathbb{C} : \Re s > 0 \}$, although strictly speaking we probably ought to keep the notation consistent and write $\mathbb{C}_0$ instead. Throughout the paper, the term *half-plane* will be used in the restricted sense of a half-plane of the type $\mathbb{C}_\theta$ for some $\theta \in \mathbb{R}$.

It should be mentioned that, by the closed graph theorem, every composition operator $\mathcal{C}_\Phi\colon \mathcal{H} \to \mathcal{H}$ is automatically bounded.

1. Results

The first question that arises naturally in connection with this problem is: For what functions Φ analytic in some half-plane $\mathbb{C}_\theta$ and mapping it into $\mathbb{C}_{1/2}$ does

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