Spectrum of the Laplacian on Asymptotically Euclidean Spaces

Harold Donnelly

1. Introduction

The Laplacian Δ for Euclidean space R^n has the following properties: (a) the essential spectrum of $-\Delta$ is $[0, \infty)$; (b) Δ has no point spectrum; and (c) Δ has no singular continuous spectrum. If $(x_1, x_2, \dots, x_n)$ are the standard global coordinates on R^n, then the *exhaustion function* $b(x) = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ satisfies (i) $|\nabla b| = 1$ for $x \neq 0$ and (ii) $\operatorname{Hess} b^2 = 2g$. Here g denotes the Euclidean metric.

Let M be a complete Riemannian manifold that admits a proper exhaustion function b. If (i) and (ii) above are satisfied in a weak or approximate sense, then we would like to show that the Laplacian Δ of M has properties similar to those of the Euclidean Laplacian. This program was started in our earlier paper [6]. Under general averaged L_2 conditions on $|\Delta b|$ and $||\nabla b| - 1|$, we showed that the essential spectrum of $-\Delta$ is $[0, \infty)$. More stringent pointwise decay conditions for $|\operatorname{Hess} b^2 - 2g|$ and $||\nabla b| - 1|$ were needed to eliminate the possibility of a point spectrum for Δ. The singular continuous spectrum was not discussed in [6].

The present paper extends the earlier work concerning the point spectrum and provides new results about the singular continuous spectrum. If M admits an exhaustion function b having Properties 2.1, then Theorem 2.3 states that Δ has no square integrable eigenfunctions. The analogous result in [6] required the stronger hypotheses $||\nabla b| - 1| \le cb^{-\varepsilon}$ and $|\operatorname{Hess} b^2 - 2g| \le cb^{-\varepsilon}$ for some $\varepsilon > 0$, whereas Properties 2.1 impose no specific decay rate on these quantities. However, Property 2.1(iv) restricts the third derivatives of b, whereas no such condition was imposed in [6]. For manifolds with nonnegative Ricci curvature, Euclidean volume growth, and quadratic curvature decay, Cheeger and Colding [3] and Colding and Minicozzi [4] constructed an exhaustion function with Properties 2.1.

The singular continuous spectrum is studied in Section 3. If b satisfies Properties 3.1 (which are more restrictive than 2.1) then Theorem 3.5 states that $-\Delta$ has no singular continuous spectrum. The asymptotically Euclidean spaces of [1] support exhaustion functions with Properties 3.1. For these spaces, the curvature may have variable sign but the curvature decay is faster than quadratic. Our treatment of the singular continuous spectrum is an application of the abstract Mourre

Received January 17, 1998. Revision received October 28, 1998.
Partially supported by NSF Grant DMS-9622709.
Michigan Math. J. 46 (1999).