Comparison of the Pluricomplex and the Classical Green Functions

Magnus Carlehed

1. Introduction

1.1. The Classical Green Function

We define the fundamental solution for the Laplacian in $\mathbf{R}^N$ as

$$p(x) = \begin{cases} \log|x| & \text{if } N = 2, \\ -|x|^{2-N} & \text{if } N \geq 3. \end{cases}$$

Let Ω be a bounded domain in $\mathbf{R}^N$ with Lipschitz boundary, and fix $y \in \Omega$. Then Ω is regular for the Dirichlet problem

$$\begin{cases} \triangle u(x) = 0 \text{ in } \Omega, \\ u(x) = -p(x-y) \text{ on } \partial\Omega; \end{cases}$$

that is, there is a function $h_y(x)$, continuous on $\bar{\Omega}$, that solves this problem. Define

$$G(x, y) = p(x-y) + h_y(x).$$

This is the *classical Green function* for the Laplacian, with pole at y. It is negative and subharmonic in Ω, harmonic in $\Omega \setminus \{y\}$, and tends to zero on $\partial\Omega$. Near y, it behaves like $p(x-y)$. Furthermore, it is symmetric, that is, $G(y, x) = G(x, y)$.

Let $U(\Omega, y)$ be the class of subharmonic functions u in Ω such that $u(\zeta) \leq p(\zeta - y) + O(1)$ when $\zeta \to y$. Then, using the classical Perron method, one can easily see that

$$G(x, y) = \sup\{u(x);\ u \in U(\Omega, y),\ u \leq 0\}.$$

Remark. In most texts, the Green function is defined to be the *negative* of our Green function.

1.2. The Pluricomplex Green Function

Let Ω be a bounded domain in $\mathbf{C}^n$. Let $V(\Omega, y)$ be the class of plurisubharmonic functions u in Ω such that $u(\zeta) \leq \log|\zeta - y| + O(1)$ when $\zeta \to y$. We define the *pluricomplex Green function* for Ω with pole in $y \in \Omega$:

$$g(x, y) = \sup\{v(x);\ v \in V(\Omega, y),\ v \leq 0\}.$$

The definition is due to Klimek [K2].

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