Orbits of Hyponormal Operators

Paul S. Bourdon

1. Introduction

We show that orbits of hyponormal operators display simple growth patterns. We then use our orbital-growth observations to prove that hyponormal operators are never supercyclic, which generalizes a result due to Hilden and Wallen [6, p. 564] and answers a question raised by Kitai [7, p. 4.5]. We also establish that every hyponormal operator is "power regular," which means that if T is a hyponormal operator on the Hilbert space H, then $\lim_n \|T^n h\|^{1/n}$ exists for every $h \in H$. That every normal operator is power regular follows from results in [2] (see also [5]).

Interest in the behavior of orbits of bounded linear operators on Hilbert space derives from the invariant subspace (subset) problem for Hilbert space operators, which is to determine whether every bounded linear operator on a separable, infinite-dimensional Hilbert space H must leave invariant some proper, nonzero, closed subspace (subset) of H. Consider, for example, the following simple proposition, well known to operator theorists.

Proposition. *Suppose that the linear operator T is a contraction on the Hilbert space H such that*

(a) *there is a nonzero vector h in H whose orbit under T $(T^n h : n = 0, 1, 2, \dots)$ has limit 0, and*
(b) *there is a vector $g \in H$ whose orbit under T is bounded away from 0.*

Then there is a proper, nonzero, closed subspace M of H that is invariant for T.

Proof. Let M be the closed linear span of $\{T^n h : n = 0, 1, 2, \dots\}$. That M is invariant for T and is nonzero is clear. We now argue by contradiction that g is not in M so that M is properly contained in H.

Suppose that g is in M; thus, there must be a sequence (p_n) of polynomials such that $(p_n(T)h)$ converges to g. Let $\varepsilon > 0$ be such that $\|T^n g\| > \varepsilon$ for all nonnegative n (here, $\|\cdot\|$ denotes the norm induced by the inner product on H). Choose the positive integer j large enough so that $\|p_j(T)h - g\| < \varepsilon/2$. We then have, for every nonnegative integer k,

Received July 31, 1996. Revision received November 19, 1996.
Research supported in part by the National Science Foundation (DMS-9401206).
Michigan Math. J. 44 (1997).