Direct Product of Free Groups as the Fundamental Group of the Complement of a Union of Lines

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1. Introduction

It is well known that the fundamental group of the complement of a complex projective algebraic curve depends on the position of its singularities [4; 6; 9]. Let $\Sigma \subset CP^2$ be a union of projective lines and let $G = \pi_1(CP^2 \setminus \Sigma)$. We ask under what conditions will G be independent of the position of the singularities of Σ. The purpose of this paper is to give such a condition. First, we define a topological invariant $\beta(\Sigma)$ for Σ. To describe β, we introduce a graph Γ that lies on the arrangement of lines Σ and connects higher singularities (multiplicity ≥ 3) of Σ. This graph in general has more than one component and is not uniquely defined. However, we show that the homotopy type of Γ is independent of our choice, and we define $\beta(\Sigma)$ to be the first Betti number of Γ. In Section 3, we prove the following theorem.

THEOREM 1. *If $\beta(\Sigma) = 0$, then $G = \pi_1(CP^2 \setminus \Sigma)$ is independent of the position of the singularities and G is a direct product of free groups.*

In Section 4, we study the fundamental group of the complement of an arrangement of six lines. An arrangement of six lines can have at most four higher singularities. In case an arrangement of six lines has three or four higher singularities, all these higher singularities must be triple points. In Section 4 we also show the following.

THEOREM 2. *For an arrangement of at most six lines, G does not depend on the position of the singularities.*

Theorems 1 and 2 together imply that: *if two arrangements of lines have the same number of lines and the same local topology, and if their complements have nonisomorphic global fundamental groups, then they must both have at least seven lines and three higher singularities.* In [4], the author gave an example of a pair of arrangements of seven lines where both have three triple points and twelve double points and where their complements have nonisomorphic global fundamental groups. We see here that in this example both the number of lines and higher singularities (and their multiplicities) are smallest possible.

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