Coherentlike Conditions in Pullbacks

Stefania Gabelli & Evan Houston

1. Introduction and Preliminaries

Let M be a (nonzero) maximal ideal of a domain T, let $k = T/M$ be the residue field, let $\phi: T \to k$ be the natural projection, and let D be a proper subring of k. Let $R = \phi^{-1}(D)$ be the domain arising from the following pullback of canonical homomorphisms:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \big\downarrow & & \big\downarrow \\ T & \xrightarrow{\ \phi\ } & k = T/M. \end{array} \tag{$\square$}$$

We use K and F to denote the quotient fields of R and D, respectively. The case $k = F$ is of particular interest; in this case, we say that the diagram $\square$ is of type $\square^*$.

The goal of this paper is to characterize certain coherentlike properties of integral domains in pullback constructions of type $\square$. In one sense, this work is a sequel to that of Brewer and Rutter [BR], in which coherence and several other properties are studied in so-called generalized $D+M$ constructions—that is, pullbacks of type $\square$ in which it is assumed that $T = k + M$. ([BR] was in turn at least partly inspired by the work of Dobbs and Papick [DP] on coherence in the classical $D + M$ construction, in which $T = k + M$ is assumed to be a valuation domain.) Our work in this more general context is partly motivated by the fact that results which hold for the $D + M$ construction do not always extend to pullbacks of type $\square$. For example, [FG, Thm. 4.2(b)] shows that the characterization of the GCD-property given in [BR, Thm. 11] requires modification, and [FG, Example 4.3] exploits pullbacks to give a counterexample to a conjecture of Anderson and Ryckaert [AR, Question 3.10].

The notion of v-finiteness figures prominently in [FG]. An ideal I of a domain R is said to be *v-finite* if $I^{-1} = J^{-1}$ for some finitely generated ideal J of R. We denote Section 2 to a study of divisoriality and v-finiteness in pullbacks of type $\square$. We show, for example, that if T is quasilocal with maximal ideal M, then each

Received December 22, 1995. Revision received July 12, 1996.
The first author was partially supported by research funds of Ministero dell'Università e della Ricerca Scientifica e Tecnologica. The second author was supported in part by funds provided by the Consiglio Nazionale delle Ricerche and by the University of North Carolina at Charlotte.
Michigan Math. J. 44 (1997).