On Entire Rational Maps in Real Algebraic Geometry

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1. Introduction and Results

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$ be real algebraic sets. A map $F: X \to Y$ is said to be *entire rational* if there exist $f_i, g_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i = 1, \ldots, m$, such that each g_i vanishes nowhere on X and

$$F = (f_1/g_1, \ldots, f_m/g_m).$$

We say X and Y are *isomorphic* to each other if there are entire rational maps $F: X \to Y$ and $G: Y \to X$ such that $F \circ G = \mathrm{id}_X$ and $G \circ F = \mathrm{id}_Y$. Let $R(X, Y)$ denote the set of all the entire rational maps from X to Y. Although the set of polynomial maps between X and Y is not an isomorphism invariant of the pair (X, Y), $R(X, Y)$ is an isomorphism invariant of the pair (X, Y). In other words, $R(X, Y)$ is independent of the embeddings of the real algebraic sets into the affine spaces (cf. [4, Chap. 3]).

In general, very little is known about entire rational maps between X and Y. In [7], Loday showed that for $n > 1$, any polynomial map from T^n to S^n is null homotopic, where T^n is the n torus $S^1 \times \cdots \times S^1$ and S^n is the standard n-sphere in $\mathbb{R}^{n+1}$. Let k and n be positive integers where k is odd and $k < 2n$. In [3], Bochnak and Kucharz showed that any entire rational map from $X \times S^{2n-k}$ to S^{2n} is null homotopic, where X is any k-dimensional nonsingular real algebraic set. The proofs of these results use algebraic K-theory. For a nice account of similar results, and for the results dealing with approximations of smooth maps by entire rational maps, we refer the reader to [1; 2; 3; 4; 7]. In all these cited results the target space is mostly the standard n-sphere S^n. In this paper, by using different and rather elementary techniques, in some cases we will prove more general results. For instance, in the statement of the Bochnak–Kucharz result we will replace the target space S^{2n} with any nonsingular real algebraic set homeomorphic to S^{2n}.

Here, a *complexification* $X_{\mathbb{C}} \subseteq \mathbb{C}P^n$ of X will mean that X is embedded in some $\mathbb{R}P^n$ and $X_{\mathbb{C}} \subseteq \mathbb{C}P^n$ is the complexification of the pair $X \subseteq \mathbb{R}P^n$. Our main theorem follows.

THEOREM 1.1. *Let X and Y be compact connected nonsingular orientable real algebraic sets of the same dimension n. Then any entire rational map*

Received April 5, 1994. Revision received November 28, 1994.
Michigan Math. J. 42 (1995).