Rigidity of Minimal Submanifolds in Spheres in Terms of Higher Fundamental Forms

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1. Preliminaries and Statement of Results

In this paper we will give a generalization of the DoCarmo–Wallach theory for rigidity and nonridigity of minimal submanifolds in spheres, introducing higher-order contact conditions for the submanifolds expressed in terms of their higher-order fundamental forms and osculating bundles.

Given an isometric immersion $F: M \to S^N$ of a Riemannian manifold M of dimension m into the Euclidean N-sphere S^N, we denote by $\beta_l(F)$, $l = 1, \dots, p_F$, the lth fundamental form of F, and by $\mathcal{O}_F^l$ the lth osculating bundle of F, all defined on a (maximal) nonempty open set D_F of M (cf. [1; 6]). For $x \in D_F$, $\beta_l(F)_x: S^l(T_xM) \to \mathcal{O}_{F;x}^l$ is a linear map of the lth symmetric power of T_xM onto the fibre $\mathcal{O}_{F;x}^l$ of $\mathcal{O}_F^l$ at x (also called the lth osculating space of F at x). Recall that $\beta_1(F) = F_*$ is defined on $D_F^1 = M$ and, for $x \in D_F^1$, the first osculating space $\mathcal{O}_{F;x}^1$ is the image of $\beta_1(F)_x$. The higher fundamental forms and osculating bundles are then defined inductively by setting

$$\begin{gathered} \beta_{l+1}(F)_x(X^0, \dots, X^l) = (\nabla_{X^0} \beta_l(F))(X^1, \dots, X^l)^{\perp_l}, \\ X^0, \dots, X^l \in T_xM, \; x \in D_F^l, \end{gathered} \tag{1}$$

where $\perp_l$ is the orthogonal projection with kernel $\mathcal{O}_{F;x}^0 \oplus \cdots \oplus \mathcal{O}_{F;x}^l$, $\mathcal{O}_{F;x}^0 = \mathbf{R} \cdot F(x)$, and D_F^{l+1} is the set of points $x \in D_F^l$ at which the image $\mathcal{O}_{F;x}^{l+1}$ of $\beta_{l+1}(F)_x$ has maximal dimension. $\beta_{p_F}(F)$ is the highest nonvanishing fundamental form, and p_F is said to be the *geometric degree* of F. We have $D_F = \bigcap_{l=0}^{p_F} D_F^l$. Finally, it is convenient to define the 0th fundamental form of F to be F itself. To be consistent with (1), we also put $\nabla_{X^0} \beta_0(F) = \beta_1(F)(X^0) = F_*(X^0)$. Note that if M and F are analytic then D_F is dense in M.

Given two isometric immersions $F: M \to S^N$ and $f: M \to S^n$, we say that *f is derived from F*, written as $f \leftharpoonup F$, if $f = A \cdot F$ for some linear map A: $\mathbf{R}^{N+1} \to \mathbf{R}^{n+1}$. If F is full (i.e., the image of F is not contained in a great sphere of S^N) then A is uniquely determined. If f is full then A is onto; f is (orthogonally) equivalent to F if ($N = n$ and) A is orthogonal.

Now let $f \leftharpoonup F$ via $f = A \cdot F$. We introduce

$$\langle f \rangle = A^{\mathsf{T}} \cdot A - I \in S^2(\mathbf{R}^{N+1}),$$

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