Extensions of Projective Varieties and Deformations, I

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0. Introduction

0.1. In this paper we deal with the following question. Let $V \subset \mathbf{P}^n$ be a projective variety; what are the obstructions for V being a hyperplane section of a projective variety $W \subset \mathbf{P}^{n+1}$? We will say that such a variety W is an *extension* of V.

Of course, if W is a cone with the base V and a vertex in $\mathbf{P}^{n+1} \setminus \mathbf{P}^n$, then V will be its hyperplane section. The point is whether we can find such a W that is not a cone. We will call such a W a *nontrivial extension* of V.

The question we have just asked has quite a long history. As early as 1909, G. Scorza proved that if V is a Veronese variety of dimension greater than 1 or a Segre variety other than $\mathbf{P}^1 \times \mathbf{P}^1 \subset \mathbf{P}^3$, then V admits no nontrivial extension. Nowadays, extensions of projective varieties have been studied by many authors (see Section 0.4 and the references in [L2]). The goal of this paper is twofold: first, to prove a result on non-extendibility of a smooth projective variety; and second, to give an interpretation of the obstruction to extendibility we use in terms of deformation theory.

0.2. PRELIMINARIES. The base field will be the field $\mathbf{C}$ of complex numbers. Let V be a smooth projective variety in $\mathbf{P}^n = \mathbf{P}(E)$, where E is an $(n+1)$-dimensional vector space. Throughout the paper $\mathbf{P}(E)$ denotes Proj Sym(E^*), so closed points of $\mathbf{P}(E)$ are lines in E. From now on, we assume that V is not contained in a hyperplane of $\mathbf{P}^n$, unless stated otherwise.

Let us state some facts known as the folklore. Consider the vector bundle (locally free sheaf) $\Gamma_V = (P^1(O_V(1)))^*$, where P^1 denotes the sheaf of principal parts of the first order. The bundle Γ_V can be included in the following exact sequences:

$$0 \to O_V(-1) \to \Gamma_V \to T_V(-1) \to 0; \tag{0.1}$$

$$0 \to \Gamma_V \to E \otimes O_V \to N_{\mathbf{P}(E)|V}(-1) \to 0. \tag{0.2}$$

Here, T_V denotes the tangent bundle of V and $N_{\mathbf{P}(E)|V}$ denotes the normal bundle of the imbedding $V \subset \mathbf{P}(E)$. The rank of Γ_V equals $\dim V + 1$; if $(\Gamma_V)_p \subset E$ is the fiber of Γ_V at the point $p \in V$ imbedded in E by the injection

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