# ON THE CENTRAL LIMIT THEOREM FOR THETA SERIES

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*To the memory of David L. Williams*

**1. Introduction.** In this paper we consider the sums

$$S_N(x) = \tfrac{1}{2} + \sum_{n=1}^{N-1} \exp(i\pi n^2 x) + \tfrac{1}{2}\exp(i\pi N^2 x)$$

for real $x$ and positive integers $N$ and study the behavior of the distribution functions $D_N(\lambda) = |\{x \in [0,1] : \lambda \le N^{-1/2}|S_N(x)|\}|$ as $N$ tends to infinity, where $\lambda$ is a non-negative real, and $|A|$ denotes the Lebesgue measure of the set $A$. One result is that for all $N$ there are constants $c_0, c_1, c_2$ such that if $0 \le \lambda \le c_0\sqrt{N}$ then $c_1(1+\lambda^4)^{-1} \le D_N(\lambda) \le c_2(1+\lambda^4)^{-1}$. We conjecture that the limit of $D_N(\lambda)$ as $N$ tends to $\infty$ exists, which would be the central limit theorem for theta series. This however appears rather difficult. We prove a somewhat related statement on the theta series

$$\theta_0(z) = \sum_{n=-\infty}^{\infty} \exp(i\pi n^2 z)$$

where $z$ is a complex number with positive imaginary part. Then

$$|\{x \in [0,1] : \lambda \le N^{-1/2}|\theta_0(x+iN^{-2})|\}|$$

converges for all $\lambda \ge 0$ as $N$ tends to infinity. The form of the limit is complicated but may be given explicitly.

The methods used to obtain these results are different from those used to treat similar questions regarding sums of the form $\sum_{k=0}^{N} \exp(i\pi n_k x)$ in place of $S_N$, where $n_k$ is a rapidly increasing sequence of integers. In the case of Hadamard gaps, for example, $n_k = 2^k$, the summands are sufficiently independent for probabilistic methods to show that their distributions converge to a normal distribution [5], [6], [8]. We rely upon a general form of the functional equation for theta series and the asymptotic expansion of $S_N$ in neighborhoods of rational points due to Fiedler, Jurkat and Körner [2]. We partition [0, 1] into subintervals on which the behavior of the sums $S_N$ and $\theta_0$ can be described by an associated function in a neighborhood of 0. The distribution function can then be expressed as a sum similar to a Riemann sum involving these associated functions.

The following notation will be used. $E(x)$ denotes $e^{i\pi x}$. Given a natural number $q$, $\sum_{h \bmod q}$ means summation over a complete residue system modulo $q$. If $g(x)$ is a complex valued measurable function of $x \in [0,1]$, and if $\lambda \ge 0$ then $D(\lambda; g) = |\{x \in [0,1] : \lambda \le |g(x)|\}|$. If $P(x, y, \ldots, z)$ is a property of variables $x, y, \ldots, z$,

$$\chi(x, y, \ldots, z; P) = \begin{cases} 1 & \text{if } P(x, y, \ldots, z) \text{ holds} \\ 0 & \text{otherwise.} \end{cases}$$

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