INVARIANT SIMPLE CLOSED CURVES ON THE TORUS

Nelson G. Markley

The behavior of the orbits of a continuous flow on the torus is potentially considerably more complicated than that of a continuous flow on a portion of the plane. Specifically, nonperiodic recursion can occur on the torus. However, if there exists a simple closed invariant but not necessarily periodic curve which is not null-homotopic, then this difficulty does not arise, because by cutting along this curve one obtains a flow on a closed annulus in the plane. Our goal is to obtain sufficient conditions for the existence of such a curve.

The main theorems (Theorems 1, 2, and 7) state that such a curve exists if there exists a positive orbit on the torus satisfying the following conditions: (a) its lift to the plane does not deviate too much from a ray with rational slope, and (b) its ω-limit set is locally connected or the fixed points in its ω-limit set are totally disconnected.

An earlier version of Theorem 2 with (b) replaced by the hypothesis that there are only finitely many fixed points in the ω-limit set appears in William O'Toole's master's thesis [4], which the author directed. I wish to thank him for his assistance in organizing and clarifying some of the arguments which are needed for both results.

Let X be a metric space and let $\mathbb{R}$ denote the real numbers. A *continuous flow* on X is a continuous mapping $\pi: X \times \mathbb{R} \to X$ such that $\pi(x,0) = x$ and

$$\pi(\pi(x,s),t) = \pi(x,s+t), \qquad \text{for all } x \in X \text{ and } s, t, \in \mathbb{R}.$$

We will suppress the π and simply write xt for $\pi(x,t)$. The *set of fixed points* of a continuous flow is defined by $F = \{x: xt = x \text{ for all } t \in \mathbb{R}\}$, and the *orbit* and *positive semi-orbit* of a point x in X are defined by

$$\mathcal{O}(x) = \{xt: t \in \mathbb{R}\} \quad \text{and} \quad \mathcal{O}^+(x) = \{xt: 5t \geq 0\}$$

respectively. The ω-*limit* set which is defined by

$$\omega(x) = \bigcap_{t \geq 0} \overline{\mathcal{O}^+(xt)} = \bigcap_{t \geq 0} \{xs: s \geq t\}^-$$

will play an important role in the sequel.

Let $(\tilde{X},p)$ be a covering space of X. Given a continuous flow π on X, there exists a unique continuous flow $\tilde{\pi}$ on $\tilde{X}$ such that

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