A SIMPLIFIED TREATMENT OF THE STRUCTURE OF SEMIGROUPS OF PARTIAL ISOMETRIES

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Preliminaries. One-parameter semigroups of partial isometries were studied in [3] and [4], where a complete structure theorem was obtained for the nilpotent case. In this note, we offer a new and simplified treatment; moreover we are able to dispense with the assumption that the semigroup is nilpotent. We begin with a brief review of our original approach.

Let S_t $(0 \leq t < \infty)$ be a strongly continuous one-parameter semigroup of partial isometries on a separable Hilbert space H. We call S_t *nilpotent* if $S_{t_0} = 0$ for some t_0; we call the smallest such t_0 the *index* of S_t, and we denote it by $i(S_t)$. If K is a separable Hilbert space and $\alpha > 0$, we denote by $L^2(K, \alpha)$ the Hilbert space of measurable K-valued functions on $[0, \alpha]$ with square-integrable K-norm. For $f \in L^2(K, \alpha)$, define

$$R_t f(x) = \begin{cases} 0 & \text{if } x < t, \\ f(x - t) & \text{if } t \leq x \leq \alpha, \end{cases}$$

with the understanding that $R_t \equiv 0$ if $t \geq \alpha$. Then R_t is a nilpotent semigroup of partial isometries, and $i(R_t) = \alpha$. We say that S_t is a *truncated shift* if it is unitarily equivalent to some R_t semigroup. For each operator A, we denote by ran A and ker A the range and null-space, respectively, of A. The statement $B \longleftrightarrow C$ means that B and C commute.

In [3], the following theorem was proved.

THEOREM A. *If* S_t *is a semigroup of partial isometries and* $i(S_t) = \alpha$, *then the following statements are equivalent:*

(a) S_t *is a truncated shift,*

(b) *the von Neumann algebra generated by the* S_t $(0 \leq t \leq \alpha)$ *is a factor,*

(c) *for each* t $(0 \leq t \leq \alpha)$, $\operatorname{ran} S_t = \ker S_{\alpha - t}$.

The difficult step turned out to be the implication (b) $\Rightarrow$ (c). This was effected by a laborious argument based on the structure of the discrete case given in [1], and involving a fairly delicate limiting argument in passing to the continuous case. Using Theorem A and the reduction theory for von Neumann algebras, we proved in [4] that each nilpotent S_t is the direct integral of truncated shifts. In the treatment given below, we avoid the difficult transition from the discrete to the continuous. The argument is almost (but not quite) self-contained. The punch line of the proof is a novel characterization of truncated shifts, which is embodied in the Lemma at the end of the proof. The argument used in the Lemma is disjoint from that used in the rest of the paper, so that the reader can have the denouement at the outset.

THEOREM. *Let* S_t *be a semigroup of partial isometries on* H. *Then*

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