ON THE DIFFERENTIABILITY OF RADEMACHER SERIES

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In this paper, we present a detailed investigation of the differentiability properties of Rademacher series. A number of authors, among them L. A. Balašov [1], J. R. McLaughlin [13], [14], and A. I. Rubinstein [16], have considered related problems. (For a brief survey of the literature, see Balašov and Rubinstein [2, pp. 748, 749].) A principal theme in the present paper concerns the category- and measure-theoretic properties of the points of differentiability of Rademacher series, and we draw on the general theory of derivatives as developed in [20] by Z. Zahorski.

In Section 2, a "zero-one" law is proved for the set of points of differentiability. Using a result of M. K. Fort [7], we show that this set is of the second category on $[0, 1)$ if and only if the series is piecewise linear. It is also shown that if a Rademacher series possesses a nonzero derivative at even one point, then the sequence of coefficients is eventually strictly monotone.

We show in Section 3 that differentiability of a Rademacher series at at least one point is sufficient to guarantee that the series is of bounded pth variation for every $p > 1$. In Section 4, a necessary and sufficient condition is obtained for a Rademacher series to be continuous in the Darboux sense (that is, for the series to carry connected sets into connected sets). It is shown that a Rademacher series that is Darboux-continuous cannot possess a derivative at any point of $[0, 1)$, except in the case where the series is piecewise linear.

Section 5 deals with series satisfying Lusin's condition (N), that is, series that map nullsets into nullsets. For Rademacher series, we show that this condition is equivalent to the preservation of measurable sets. It is also shown that differentiability at even one point is sufficient to imply condition (N).

The Dini derivates of Rademacher series are examined in detail in Section 6. We show that on a residual set in $[0, 1)$, the upper and lower derivates are infinite and of opposite sign. This set is further shown to be of full measure in $[0, 1)$ unless the series has a derivative almost everywhere.

In the last two sections, we treat the problem of determining necessary and sufficient conditions for a Rademacher series to be of bounded variation. We show, in particular, that a Rademacher series is differentiable almost everywhere on $[0, 1)$ if and only if it is of bounded variation. This result answers a question raised by McLaughlin in [13], and it is an analogue for the Walsh system of the corresponding result for lacunary trigonometric series (McLaughlin [12]). Explicit representations for the total variation of a Rademacher series are also obtained. Finally, we show that if a Rademacher series possesses a nonzero derivative almost everywhere in $[0, 1)$, there exists a perfect set of positive measure on which the series is strictly monotone.

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