CONVEXITY PROPERTIES OF OPERATOR RADII ASSOCIATED WITH UNITARY ρ-DILATIONS

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1. INTRODUCTION

For $\rho > 0$, let $\mathscr{C}_\rho$ denote the class of bounded linear operators T on a Hilbert space $\mathscr{H}$ whose powers admit a representation

$$T^n h = \rho P U^n h \qquad (h \in \mathscr{H};\ n = 1, 2, \cdots),$$

where U is a unitary operator (called a *unitary ρ-dilation*) on some Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ as a subspace, and where P is the projection from $\mathscr{K}$ to $\mathscr{H}$. Intrinsic characterizations of operators of class $\mathscr{C}_\rho$ were given by B. Sz.-Nagy and C. Foiaş [6]. Later, J. A. R. Holbrook [3] and J. P. Williams [7] introduced the concept of the operator radius $w(\rho)$ of an operator T, relative to $\mathscr{C}_\rho$. The operator radius is defined by the formula

$$w(\rho) = w(\rho; T) = \inf\{\gamma: \gamma > 0,\ \gamma^{-1} T \in \mathscr{C}_\rho\}.$$

It is known that $w(1)$ coincides with the norm $\|T\|$ while $w(2)$ is simply the numerical radius

$$w(2) = \sup\{|(Th, h)|: \|h\| = 1\}.$$

Holbrook [3], [4] investigated basic properties of $w(\rho)$. Among other things, he showed that $w(\rho)$ is a nonincreasing function of ρ, that

$$w(1) \leq \rho \cdot w(\rho) \leq (2\rho' - \rho) \cdot w(\rho') \qquad (\rho \leq \rho'),$$

and that $w(\infty) = \lim_{\rho \to \infty} w(\rho)$ coincides with the spectral radius of T.

Further, Holbrook [4] proved the convexity of $w(\rho)$ on $(0, 1)$, and he asked whether $w(\rho)$ is convex on the whole interval $(0, \infty)$. Our main purpose in this paper is to prove that $\log w(\rho)$ is convex on $(0, \infty)$.

In Section 2, using function-theoretic methods, we shall show that $\log w(\rho)$ and $\log\{(e^\xi + 1) w(e^\xi + 1)\}$ are convex on $(0, \infty)$ and $(-\infty, \infty)$, respectively. Incidentally, we point out the reciprocity law

$$\rho \cdot w(\rho) = (2 - \rho) \cdot w(2 - \rho) \qquad (0 < \rho < 2),$$

which has hitherto been overlooked. As a consequence of convexity, we show that $\rho \cdot w(\rho)$ is decreasing on $(0, 1)$ and increasing on $(1, \infty)$.

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