# ZEROS OF LIPSCHITZ FUNCTIONS ANALYTIC IN THE UNIT DISC

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## 1. INTRODUCTION

Let D denote the open unit disc in the complex plane, and let $\overline{D}$ denote its closure. Let Lip $\alpha$ be the class of functions f analytic in D and satisfying a Lipschitz condition of order $\alpha$,

$$(1.1)\qquad |f(z) - f(z')| \leq C\,|z - z'|^{\alpha}.$$

L. Carleson [1] gave a necessary and sufficient condition for a closed set $E \subset \partial D = \overline{D} \setminus D$ to be the zero set of a function $f \in$ Lip $\alpha$. If $\rho(z, E)$ denotes the Euclidean distance from z to E, then evidently (1.1) implies that

$$\log|f(z)| \leq \alpha \log \rho(z, E) + \log C,$$

and consequently

$$(1.2)\qquad \int_{-\pi}^{\pi} \log \rho(e^{i\theta}, E)\, d\theta > -\infty,$$

by a well-known theorem of F. Riesz. Conversely, Carleson showed that if (1.2) holds, then there exists an outer function f such that $f(e^{i\theta}) = 0$ if and only if $e^{i\theta} \in E$, and that for each integer $m > 0$ the function f can be constructed so that it belongs to the class $A^m$ of functions that are analytic in D and whose first m derivatives are continuous in $\overline{D}$. W. P. Novinger [3] and we extended this result independently by showing that f can be constructed so that it belongs to the class $A^\infty = \bigcap_{m=1}^{\infty} A^m$. Also, a result has recently been proved by Carleson and S. Jacobs that implies the following: if $f \in A = A^0$, if f is an outer function, and if $|f(e^{i\theta})|$ has 2m continuous derivatives as a function of $\theta$, then $f \in A^m$. This theorem yields an easy proof of the extension of Carleson's theorem discussed above.

In this paper, we solve the analogous problem for zero sets in $\overline{D}$. In the following, Z denotes a closed subset of $\overline{D}$ such that $Z \cap D$ is countable. To each element of $Z \cap D$ we assign a multiplicity, and we let $\{z_j\}_{j=1}^{\infty}$ be an enumeration of $Z \cap D$ with each element of $Z \cap D$ appearing in the sequence a number of times equal to its multiplicity. Also, $\rho(z)$ $(z \in \overline{D})$ denotes the Euclidean distance from z to Z.

THEOREM. *In order that for some* $\alpha$ $(0 < \alpha \leq 1)$ *there exist a function* $f \in$ Lip $\alpha$ *whose zero set is* Z (*counting multiplicities*), *it is necessary that*

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