# INVARIANT SUBSPACES FOR ANALYTICALLY COMPACT OPERATORS

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Let T be a bounded operator on a Banach space, and let P be a nonzero polynomial. The theorem of A. R. Bernstein and A. Robinson [2] says that if P(T) is compact, then T has a nontrivial invariant subspace. An easy generalization is that if f is analytic in a neighborhood of the spectrum $\sigma(T)$ and f(T) is compact, while f is not identically zero, then T has a nontrivial invariant subspace. A related result of W. B. Arveson and J. Feldman [1] says that if a nonzero compact operator C is contained in the uniform closure $\mathscr{A}(T)$ of the polynomials in T, and if in addition T is quasi-nilpotent, then T has a nontrivial invariant subspace. (This was proved in [1] for operators on Hilbert space; several authors ([3], [4], [5]) have extended it to arbitrary normed spaces.) One might hope to generalize all these results by eliminating the hypothesis of quasi-nilpotence in the last theorem. Although we have not done so, we have produced another fragment of evidence in this direction, namely, the following.

THEOREM. *Let* T *be a bounded linear operator on a Banach space. Suppose that some analytic expression in* T *is a nonzero compact operator* C. *Then* T *has a nontrivial invariant subspace.*

Before we go into the proof, we should, of course, say what we mean by "analytic expression." To begin with, if f is analytic in a neighborhood of $\sigma(T)$, we call f(T) a *basic analytic expression* in T. Also, if $a_0, a_1, \cdots$ is a sequence of complex numbers such that $\sum a_n T^n$ converges in norm, we call this sum a *basic analytic expression*. By a *(general) analytic expression* we mean an element of the ring generated by the basic analytic expressions. A general analytic expression is thus an operator of the form $p(A_1, \cdots, A_n)$, where p is a polynomial and $A_1, \cdots, A_n$ are basic analytic expressions. (We could allow greater breadth to the concept of analytic expression, as will become clear in the course of our argument, but for the sake of simplicity we delimit it as indicated. We shall comment on this at the end of the paper.)

Note that all the analytic expressions belong to the uniformly closed algebra $\mathscr{B}(T)$ generated by T together with the operators $(\lambda - T)^{-1}$ $(\lambda \notin \sigma(T))$. The algebra $\mathscr{B}(T)$ is generally larger than $\mathscr{A}(T)$, although it coincides with $\mathscr{A}(T)$ if $\sigma(T)$ happens to be a single point.

We shall need two lemmas. The first of these is a generalization of Abel's theorem to series in a Banach space.

LEMMA 1. *Let* $A_0, A_1, A_2, \cdots$ *be elements of a Banach space. Suppose that the series* $\sum A_n$ *converges. Let* $0 < \theta < \pi/2$, *and let* $S_\theta$ *be the angular region*

$$S_\theta = \{z: |z| < 1, -\theta < \arg(1 - z) < \theta\}.$$

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