A POWER-BOUNDED OPERATOR THAT IS NOT POLYNOMIALLY BOUNDED

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Foguel has constructed an operator with uniformly bounded powers that is not similar to a contraction [1]. This counterexample answers a question asked by B. Sz.-Nagy [6]. Halmos has given a less computational version of Foguel's arguments in [2]. The purpose of this note is to reexamine Foguel's operator and to show that it is not polynomially bounded; from this it follows that Foguel's operator is not a counterexample to the conjecture that each polynomially bounded operator is similar to a contraction.

An operator T on a Hilbert space is said to be *polynomially bounded* if there exists a constant K such that

$$(*) \qquad \|\mathscr{P}(T)\| \leq K \sup\{|\mathscr{P}(z)|: |z| \leq 1\}$$

for every polynomial $\mathscr{P}$. Another way of describing this condition is to say that the unit disc is a K-spectral set for T. It is a well-known result, due to von Neumann, that the unit disc is a 1-spectral set for each contraction. The elegant proofs of this result proceed by reduction to the case of a unitary operator [4], [5]. Now, if T is similar to a contraction C (that is, if $T = S^{-1}CS$ with $\|C\| \leq 1$), then $\mathscr{P}(T) = S^{-1}\mathscr{P}(C)S$ for each polynomial $\mathscr{P}$. Thus it follows from von Neumann's theorem that T is polynomially bounded, with $K = \|S^{-1}\| \cdot \|S\|$. Therefore an operator that is not polynomially bounded is not similar to a contraction.

An operator T is said to be a *moment operator* if for each pair of vectors x and y there exists a complex-valued function g on $[0, 2\pi]$, of finite variation, such that

$$\langle T^n x, y \rangle = \int_0^{2\pi} e^{int} dg(t) \qquad \text{for } n = 0, 1, 2, \cdots.$$

LEMMA. *An operator is polynomially bounded if and only if it is a moment operator.*

Proof. If T is polynomially bounded, consider $\langle \mathscr{P}(T)x, y \rangle$ as a function of $\mathscr{P}$, and apply the Schwarz inequality, (*), and the maximum-modulus, Hahn-Banach, and Riesz representation theorems to show the existence of a g such that

$$\langle \mathscr{P}(T)x, y \rangle = \int \mathscr{P} dg.$$

(The argument has been used in another context [3].)

To prove the converse, note that if T is a moment operator and $|\mathscr{P}(z)| \leq 1$ for $|z| \leq 1$, then

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